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Mathematical Option Pricing

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Further Results in Stochastic Analysis

1.1 The Martingale Representation Theorem for Brownian Motion

Let $W_t, t \geq 0$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_t$ be the natural filtration: $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$.

Theorem 1.1. *Let $T > 0$ and suppose that $X \in L_2(\Omega, \mathcal{F}_T, P)$. Then there exists an adapted process g_t such that $E\int_0^T g^2(s)ds < \infty$ and*

$$X = EX + \int_0^T g(s)dW_s. \tag{1.1}$$

The proof follows from the Lemmas below. First, recall that a subset $\mathcal{D}$ of $L_2(\Omega, \mathcal{F}_T, P)$ is *dense* if for every $X \in L_2(\Omega, \mathcal{F}_T, P)$ we have $\mathcal{D} \cap B \neq \emptyset$ for every neighbourhood B of X. In particular, there exists a sequence $X_n \in \mathcal{D}$ such that $X_n \to X$.

Lemma 1.2. *Theorem 1.1 holds if the representation (1.1) holds for every X in some dense subset $\mathcal{D}$ of $L_2(\Omega, \mathcal{F}_T, P)$.*

Proof: Let $X \in L_2(\Omega, \mathcal{F}_T, P)$ and take $X_n \in \mathcal{D}, X_n \to X$ as described above. Then $EX_n \to EX$ and there exist integrands g_n such that

$$X_n = EX_n + \int_0^T g_n(s)dW_s. \tag{1.2}$$

Taking $\tilde{X}_n = X_n - EX_n$ we have the Ito isometry

$$E(\tilde{X}_n - \tilde{X}_m)^2 = E\int_0^T (g_n(s) - g_m(s))^2 ds \tag{1.3}$$

Since $\tilde{X}_n$ is convergent, it is a Cauchy sequence, and hence from (1.3) the sequence g_n is convergent in $L_2(\Omega \times [0,T], dP \times dt)$. Thus there exists g such that

$$E\int_0^T (g_n(s) - g(s))^2 ds \to 0 \quad \text{as } n \to \infty$$

and (1.1) holds with this integrand g. ♢

Let $\mathcal{D}_T$ be the subset of $L_2(\Omega, \mathcal{F}_T, P)$ consisting of random variables X of the form $X = h(W_{t_1}, W_{t_2}, \ldots W_{t_n})$, where n is an integer, h is a bounded continuous function from R^n to R, and $0 \leq t_1 < \ldots < T_n \leq T$. The proof of the following result is an elegant application of the martingale convergence theorem. See Øksendal [7], Lemma 4.3.1.

Lemma 1.3. *$\mathcal{D}_T$ is dense in $L_2(\Omega, \mathcal{F}_T, P)$.*

To prove the Theorem, it remains to show that any $X \in \mathcal{D}_T$ has the representation property, and this we can show by a direct argument. In the following, we take $n = 2$; the extension to $n > 2$ is obvious. First, a fact about conditional expectation.

Lemma 1.4. *Let X, Y be random variables taking values in R^n, R^m respectively, on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and suppose that X is independent of $\mathcal{G}$ while Y is $\mathcal{G}$-measurable. Then for any measurable function $f : R^{n+m} \to R$ such that $E|f(X,Y)| < \infty$, we have*

$$E[f(X,Y)|\mathcal{G}] = b(Y),$$

where

$$b(y) = \int_{R^n} f(x,y)\mu_X(dx).$$

Here μ_X is the distribution of X, the measure on the Borel sets $\mathcal{B}^n$ of R^n defined by $\mu_X(B) = P(X \in B)$ for $B \in \mathcal{B}^n$.

PROOF: We have to show that for all bounded real-valued $\mathcal{G}$-measurable random variables Z we have

$$E[Zf(X,Y)] = E[Zb(Y)].$$

Let $\mu_{X,Y,Z}$ be the distribution of the R^{n+m+1}-valued r.v. (X, Y, Z). Since X is independent of $\mathcal{G}$, the random variables X and (Y, Z) are independent, so that $\mu_{X,Y,Z}(dx, dy, dz) = \mu_X(dx)\mu_{Y,Z}(dy, dz)$. Hence

$$\begin{aligned} E[Zf(X,Y)] &= \int zf(x,y)\mu_{X,Y,Z}(dx,dy,dz) \\ &= \int z\left(\int f(x,y)\mu_X(dx)\right)\mu_{Y,Z}(dy,dz) \\ &= \int zb(y)\mu_{Y,Z}(dy,dz) \\ &= E[Zb(Y)]. \end{aligned}$$

Lemma 1.5. *Let $h : R^2 \to R$ be a bounded continuous function and let t_1, t_2, t satisfy $0 \le t_1 \le t \le t_2$. Then*

$$E[h(W_{t_1}, W_{t_2})|\mathcal{F}_t] = v_1(t, W_{t_1}, W_t),$$

where

$$v_1(t,x,y) = \int h(x,z)\frac{1}{\sqrt{2\pi(t_2-t)}}e^{(z-y)^2/2(t_2-t)}dz. \tag{1.4}$$

PROOF: Writing $h(W_{t_1}, W_{t_2}) = h(W_{t_1}, (W_{t_2} - W_t) + W_t)$, this follows immediately from Lemma 1.5, on taking $X = W_{t_2} - W_{t_1}, Y = (W_{t_1}, W_t) \in R^2$ and $f(x,y) = h(y_1, x + y_2)$, and recalling that $X \sim N(0, t_2 - t)$. ♢

Lemma 1.6. *The random variable $X = h(W_{t_1}, W_{t_2})$, as defined in Lemma 1.5, has the representation property.*

PROOF: It can be checked directly from (1.4) that the function v_1 satisfies

$$\frac{\partial v_1}{\partial t}(t,x,y) + \frac{1}{2}\frac{\partial^2 v_1}{\partial y^2}(t,x,y) = 0$$

and $v_1(T, x, y) = h(x, y)$. Hence by the Ito formula

$$h(W_{t_1}, W_{t_2}) = v_1(T, W_{t_1}, W_{t_2}) = v_1(t_1, W_{t_1}, W_{t_1}) + \int_{t_1}^{t_2} \frac{\partial v_1}{\partial y}(s, W_{t_1}, W_s)dW_s, \tag{1.5}$$

and we know from Lemma 1.5 that $v_1(t_1, W_{t_1}, W_{t_1}) = E[h(W_{t_1}, W_{t_2})|\mathcal{F}_{t_1}]$. Now define $v_0(t_1, x) = v_1(t_1, x, x)$, and, for $t < t_1$

$$v_0(t, x) = \int v_0(t_1, z) \frac{1}{\sqrt{2\pi(t_1 - t)}} e^{(z-x)^2/2(t_1-t)} dz. \tag{1.6}$$

As above, we have

$$\frac{\partial v_0}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 v_1}{\partial x^2}(t, x) = 0,$$

and the Ito formula gives

$$v_0(t_1, W_{t_1}) = v_1(t_1, W_{t_1}, W_{t_1}) = v_0(0, 0) + \int_0^{t_1} \frac{\partial v_0}{\partial y}(s, W_s) dW_s. \tag{1.7}$$

From (1.5),(1.7) we now see that

$$h(W_{t_1}, W_{t_2}) = v_0(0, 0) + \int_0^{t_2} g(s) dW_s,$$

where

$$g(s) = \begin{cases} (\partial v_0/\partial y)(s, W_s), & s < t_1 \\ (\partial v_1/\partial y)(s, W_{t_1}, W_s), & t_1 \leq s < t_2 \end{cases},$$

and that

$$v_0(0, 0) = E[h(W_{t_1}, W_{t_2})].$$

This completes the proof.

1.2 Changes of Measure

1.2.1 Normal distributions

A random variable X is normally distributed, written $X \sim N(\mu, \sigma^2)$, if its characteristic function ψ takes the form

$$\psi_\mu(u) = Ee^{iuX} = \exp\left(iu\mu - \frac{1}{2}u^2\sigma^2\right). \tag{1.8}$$

This corresponds to the density function ϕ given by

$$\phi_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right).$$

μ and σ are the mean and standard deviation respectively. (σ is fixed in the following and so is not included in the notation.)

If $X \sim N(\mu, \sigma^2)$ then for any bounded function f,

$$E[f(X)] = \int f(x)\phi_\mu(x)dx,$$

For any ν we can trivially write this as

$$E[f(X)] = \int f(x) \frac{\phi_\mu(x)}{\phi_\nu(x)} \phi_\nu(x) dx, \tag{1.9}$$

and we find that

$$\frac{\phi_\mu(x)}{\phi_\nu(x)} = \exp\left(\frac{1}{\sigma^2}(\mu - \nu)x - \frac{1}{2\sigma^2}(\mu^2 - \nu^2)\right). \tag{1.10}$$

Let us denote by Λ the random variable $\Lambda = \phi_\mu(X)/\phi_\nu(X)$. We find that

- $\Lambda > 0, \;\; E_\nu[\Lambda] = 1$
- $E_\mu[f(X)] = E_\nu[f(X)\Lambda]$, where E_μ denotes integration wrt $N(\mu, \sigma^2)$

To see the first of these, take $f(x) \equiv 1$ in (1.9), or use (1.10) and the fact that if $X \sim N(\nu, \sigma^2)$ then

$$Ee^X = e^{\nu + \frac{1}{2}\sigma^2}.$$

We can thus flip between E_μ and E_ν by introducing Λ, the *likelihood ratio* or *Radon-Nikodym derivative*. In most applications, $\nu = 0$.

1.2.2 A General Setting

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and Λ be a r.v. such that $\Lambda \geq 0$ a.s. and $E\Lambda = 1$. Then we can define a measure Q on $(\Omega, \mathcal{F})$ by

$$QF = \int_F \Lambda dP, \;\; F \in \mathcal{F}. \tag{1.11}$$

Λ is often written dQ/dP and is the Radon-Nikodym derivative of Q wrt P. Note that $PF = 0 \Rightarrow QF = 0$; we say that Q is *absolutely continuous wrt* P, written $Q \ll P$. The *Radon-Nikodym theorem* states that any Q that is absolutely continuous wrt P can be written as (1.11) for some Λ. If $\Lambda > 0$ a.s. then P is absolutely continuous wrt Q, with RN derivative $dP/dQ = 1/\Lambda$. In this case P and Q are said to be *equivalent*, written $P \sim Q$. Measures P and Q are equivalent if and only if they have *the same null sets*: $PF = 0 \Leftrightarrow QF = 0$.

1.2.3 Conditional Expectations

Let X be an integrable r.v. and $\mathcal{G}$ a sub-sigma-field of $\mathcal{F}$. The conditional expectation of X given $\mathcal{G}$ is the unique $\mathcal{G}$-measurable r.v., denoted $E[X|\mathcal{G}]$ such that

$$\int_G X dP = \int_G E[X|\mathcal{G}] dP.$$

Key properties:

1. $E[X|\mathcal{G}] = X$ if X is $\mathcal{G}$-measurable
2. $E[X|\mathcal{G}] = EX$ if X is independent of $\mathcal{G}$
3. $E[YX|\mathcal{G}] = YE[X|\mathcal{G}]$ if Y is $\mathcal{G}$-measurable
4. For $\mathcal{H} \subset \mathcal{G}$, $E[X|\mathcal{H}] = E[E[X|\mathcal{G}]|\mathcal{H}]$. In particular, $EX = E(E[X|\mathcal{G}])$ for any sub-σ-field $\mathcal{G}$.

Existence of $E[X|\mathcal{G}]$ follows from the Radon-Nikodym theorem. Indeed, the formula $Q(A) = \int_A X dP$ defines a measure on $(\Omega, \mathcal{G})$ that is absolutely continuous wrt P', the restriction of P to $\mathcal{G}$. Hence there exists a $\mathcal{G}$-measurable function Λ such that $Q(A) = \int_A \Lambda dP'$.

The following result will be needed in Section 1.3.3 below.

Lemma 1.7. *Suppose $X, X_1, X_2, \ldots$ is a sequence of integrable random variables such that $X_n \to X$ in L_1. Then for any σ-field $\mathcal{G}$, $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ in L_1.*

PROOF: First we show that if Y is any integrable r.v. then

$$|E[Y|\mathcal{G}]| \leq E[|Y|\,|\mathcal{G}] \text{ a.s.} \tag{1.12}$$

Indeed, denoting as usual $Y^+ = \max(Y, 0)$ and $Y^- = Y^+ - Y$, we have

$$E[Y|\mathcal{G}]^+ = E[Y^+ - Y^-|\mathcal{G}]^+ \leq E[Y^+|\mathcal{G}]^+ = E[Y^+|\mathcal{G}]$$

and

$$E[Y|\mathcal{G}]^- = E[-Y|\mathcal{G}]^+ \le E[(-Y)^+|\mathcal{G}] = E[Y^-|\mathcal{G}],$$

from which (1.12) follows. Now if $X_n \to X$ in L_1 then using (1.12)

$$\begin{aligned} E\,|E[X_n|\mathcal{G}] - E[X|\mathcal{G}]| &= E\,|E[X_n - X|\mathcal{G}]| \\ &\le E\,(E[|X_n - X|\,|\mathcal{G}]) \\ &= E|X_n - X| \;\to 0. \end{aligned}$$

Conditional expectation under change of measure

If P, Q are measures on $(\Omega, \mathcal{F})$ such that $Q \ll P$ with RN derivative $\Lambda = dQ/dP$, and $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ then

$$E_Q[X|\mathcal{G}] = \frac{E[X\Lambda|\mathcal{G}]}{E[\Lambda|\mathcal{G}]} \quad \text{a.s. } Q \tag{1.13}$$

To see this, calculate $E[X\Lambda|\mathcal{G}]$ by taking a set $G \in \mathcal{G}$ and using the above properties of conditional expectation. We get

$$\begin{aligned} \int_G E[X\Lambda|\mathcal{G}]dP &= \int_G X\Lambda dP \\ &= \int_G XdQ \\ &= \int_G E_Q[X|\mathcal{G}]dQ \\ &= \int_G E_Q[X|\mathcal{G}]\Lambda dP \\ &= \int_G E_Q[X|\mathcal{G}]E[\Lambda|\mathcal{G}]dP \end{aligned}$$

Thus $\int_G ZdP = 0$ for all $G \in \mathcal{G}$, where $Z = E[X\Lambda|\mathcal{G}] - E_Q[X|\mathcal{G}]E[\Lambda|\mathcal{G}]$ is a $\mathcal{G}$-measurable random variable. Hence $Z = 0$ a.s. This gives (1.13) on noting that, by definition, the set $\{\omega : E[\Lambda|\mathcal{G}] = 0\}$ has Q-measure 0.

Changes of measure and martingales

Take a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t, t \in [0, T])$. Assume for convenience that $\mathcal{F} = \mathcal{F}_T$, and suppose there is another measure Q, defined by $dQ/dP = \Lambda$, where Λ is a non-negative r.v. with $E\Lambda = 1$. An adapted process (X_t) is a *martingale* (under measure P) if it is integrable and for $s \le t$

$$X_s = E[X_t|\mathcal{F}_s] \quad \text{a.s.}$$

The main result we need is this: *a process Y_t is a Q-martingale if and only if the process $X_t = Y_t\Lambda_t$ is a P-martingale, where $\Lambda_t = E[\Lambda|\mathcal{F}_t]$*. This follows from (1.13). Indeed, for $s < t$ we have

$$\begin{aligned} E_Q[Y_t|\mathcal{F}_s] &= \frac{E[Y_t\Lambda|\mathcal{F}_s]}{E[\Lambda|\mathcal{F}_s]} \\ &= \frac{E[Y_t\Lambda_t|\mathcal{F}_s]}{\Lambda_s} \end{aligned}$$

If Y_t is a Q-martingale the left-hand side is equal to Y_s, so that $Y_t\Lambda_t$ is a martingale, while if $Y_t\Lambda_t$ is a martingale then the right-hand side is equal to Y_s, showing that Y_t is a Q-martingale.

A process X_t is a *local martingale* if there exists a sequence of stopping times τ_n such that $\tau_n \to \infty$ a.s. and for each n the process $X_t^n = X_{t\wedge\tau_n}$ is a martingale. It is also true that *a process Y_t is a Q-local martingale if and only if the process $X_t = Y_t\Lambda_t$ is a P-local martingale.* Exercise: show this.

1.3 The Lévy characterization of Brownian Motion

1.3.1 Quadratic variation of Brownian motion

Let W_t be a Brownian motion process and let T be a fixed time. For $n = 1, 2, \dots$ let $\{t_i^n, i = 0..k_n\}$ be an increasing sequences of times with $t_0^n = 0, t_{k_n}^n = T$. Denote $\Delta W_i = W_{t_{i+1}^n} - W_{t_i^n}, \Delta t_i = t_{i+1}^n - t_i^n$ and $S_n = \sum_i \Delta W_i^2$. Note that the r.v. ΔW_i are independent with $E\Delta W_i = 0, E\Delta W_i^2 = \Delta t_i$. Hence that $ES_n = T$ and

$$ES_n^2 = 2\sum_i \Delta t_i^2 + T^2. \tag{1.14}$$

The latter follows from a short calculation using the fact that if $X \sim N(0, \sigma^2)$ then $EX^4 = 3\sigma^4$. From (1.14),

$$\begin{aligned} \text{var}(S_n) &= E(S_n - T)^2 \\ &= 2\sum_i \Delta t_i^2 \\ &\le 2\max_i\{\Delta t_i\}\sum_i \Delta T_i \\ &= 2T\max_i\{\Delta t_i\}. \end{aligned} \tag{1.15}$$

Hence $S_n \to T$ in L_2 as $n \to \infty$ as long as $\max_i\{\Delta t_i\} \to 0$.

Let us now specialize to the case $t_i^n = i/2^n$. From (1.15) and the Chebyshev inequality, for any $\epsilon > 0$

$$P[|S_n - T)| > \epsilon] \le \frac{2T2^{-n}}{\epsilon^2}.$$

Taking $\epsilon = 1/n$ we find that

$$\sum_n P\left[|S_n - T)| > \frac{1}{n}\right] \le \sum_n 2Tn^2 2^{-n} < \infty$$

Hence by the Borel-Cantelli lemma we have

$$P\left[|S_n - T| > \frac{1}{n} \quad \text{infinitely often}\right] = 0,$$

showing that $S_n \to T$ almost surely. Thus for each $T > 0$ the quadratic variation $\mathrm{QV}(T)$ is equal to the deterministic function $\mathrm{QV}(T) = T$.

Suppose now that X_t is a continuous process with sample paths of bounded variation, i.e.

$$\sup_n \sum_i \left| X_{t_{i+1}^n} - X_{t_i^n} \right| < \infty \quad a.s.$$

For example, any process of the form $X_t = \int_0^t \phi(s)ds$ with integrable ϕ satisfies this. Let us compute the quadratic variation of $Y_t = W_t + X_t$. We have

$$\begin{aligned} \sum_i (Y_{t_{i+1}^n} - Y_{t_i^n})^2 &= \sum_i (W_{t_{i+1}^n} - W_{t_i^n} + X_{t_{i+1}^n} - X_{t_i^n})^2 \\ &= \sum_i \Delta W_i^2 + \sum_i \Delta X_i^2 + 2\sum_i \Delta W_i \Delta X_i \end{aligned}$$

where $\Delta W_i = W_{t_{i+1}^n} - W_{t_i^n}$ etc. The first term converges to T and the second and third converge to 0: for the third term,

$$\sum_i (W_{t_{i+1}^n} - W_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) \leq \max_i |W_{t_{i+1}^n} - W_{t_i^n}| \sum_i |X_{t_{i+1}^n} - X_{t_i^n}|.$$

The sum on the right is bounded and the "max" converges to zero because W_t is a continuous function. A similar argument applies to the second term.

We have shown that the quadratic variation of W and Y are the same: the quadratic variation of W is not altered by adding a bounded variation perturbation to the sample path.

1.3.2 Quadratic variation of continuous martingales

We can't treat this subject in complete detail here; see [8] pages 52-55 or [2]. Let M_t be a martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Because of the martingale property,

$$E[(M_t - M_s)^2 | \mathcal{F}_s] = E[M_t^2 + M_s^2 - 2M_tM_s | \mathcal{F}_s] = E[M_t^2 - M_s^2 | \mathcal{F}_s]. \tag{1.16}$$

and hence with the notation above

$$E\left[\sum_i (M_{t_{i+1}^n} - M_{t_i^n})^2\right] = E\left[\sum_i E\left((M_{t_{i+1}^n} - M_{t_i^n})^2 \middle| \mathcal{F}_{t_i^n}\right)\right] = EM_T^2, \tag{1.17}$$

using (1.16). This suggests that the left-hand side has a limit as $n \to \infty$, the *quadratic variation* of (M_t).

When (M_t) is Brownian motion we have from (1.16) for $t > s$

$$\begin{aligned} E[M_t^2 | \mathcal{F}_s] &= E[M_t^2 - M_s^2 | \mathcal{F}_s] + M_s^2 \\ &= E[(M_t - M_s)^2 | \mathcal{F}_s] + M_s^2 \\ &= t - s + M_s^2. \end{aligned}$$

Hence the process $M_t^2 - t$ is a martingale. The general situation is as follows.

Theorem 1.8. *Let M_t be a continuous local martingale. Then there is a unique continuous increasing process, denoted $[M]_t$, such that $M_t^2 - [M]_t$ is a local martingale. $[M]_t$ is the quadratic variation of M_t: it is the almost sure limit of approximating sums as in (1.17) taken along suitable sequences (t_i^n).*

We call a process X_t a *semimartingale* if it can be expressed as $X_t = M_t + A_t$ where M_t is a martingale and A_t is a process whose sample paths have bounded variation. X_t is a *continuous semimartingale* if both M_t and A_t have continuous sample paths, and a *local semimartingale* if M_t is a local martingale. The existence of $[M]_t$ gives us an Ito formula for continuous local semimartingales, analogous to the usual Ito formula for Brownian motion.

Theorem 1.9. *Let X_t be a continuous local smimartingale and f a $C^{1,2}$ function. Then*

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} d[M]_t \tag{1.18}$$

1.3.3 The Lévy characterization

Theorem 1.10. *Let M_t be a continuous local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and suppose that $[M]_t = t,\ t \geq 0$. Then M_t is an $\mathcal{F}_t$-Brownian motion.*

PROOF: Suppose M_t is a continuous local martingale with $[M]_t = t$ and take $f(t, x) = \exp(iux + u^2t/2)$. By applying (1.18) to the real and imaginary parts of f you can check that (1.18) is also valid for complex functions. We obtain

$$df(t, M_t) = \frac{1}{2}u^2 f(t, M_t)dt + iuf(t, M_t)dM_t - \frac{1}{2}u^2 f(t, M_t)d[M]_t,$$

so that $f(t, M_t)$ is a local martingale if $[M]_t = t$. Thus for $t > s$ we have

$$E\left[e^{iuM_{t\wedge\tau_n} + \frac{1}{2}u^2 t\wedge\tau_n} \middle| \mathcal{F}_s \right] = e^{iuM_{s\wedge\tau_n} + \frac{1}{2}u^2 s\wedge\tau_n}, \tag{1.19}$$

where τ_n is a sequence of localizing times. Now the sequence $\exp(iuM_{s\wedge\tau_n} + \frac{1}{2}u^2(s\wedge\tau_n))$ is bounded and converges almost surely (and hence in L_1) to $\exp(iuM_s + \frac{1}{2}u^2 s)$. By Lemma 1.7, the conditional expectation in (1.19) converges in L_1 to the conditional expectation of the limit, and we conclude that

$$E\left[e^{iuM_t + \frac{1}{2}u^2 t} \middle| \mathcal{F}_s \right] = e^{iuM_s + \frac{1}{2}u^2 s},$$

or, equivalently,

$$E\left[e^{iu(M_t - M_s)} \middle| \mathcal{F}_s \right] = e^{-\frac{1}{2}u^2(t-s)}. \tag{1.20}$$

Now let Y be any $\mathcal{F}_s$-measurable random variable, and ψ_Y be the characteristic function of Y. Then by Property (3) of conditional expectation (see Section 1.2.3 above) the joint characteristic function of Y and $M_t - M_s$ is

$$\begin{aligned}
\psi_{Y,M_t-M_s}(v,u) &= E\left[e^{i(vY + u(M_t - M_s))} \right] \\
&= E\left[e^{ivY} e^{iu(M_t - M_s)} \right] \\
&= E\left[e^{ivY} E\left[e^{iu(M_t-M_s)} \middle| \mathcal{F}_s \right] \right] \\
&= E\left[e^{ivY} \right] e^{-\frac{1}{2}u^2(t-s)} \\
&= \psi_Y(v)\psi_{M_t-M_s}(u).
\end{aligned}$$

Thus Y and $(M_t - M_s)$ are independent, implying – since Y is arbitrary – that $(M_t - M_s)$ is independent of $\mathcal{F}_s$. From (1.20), $(M_t - M_s)$ is normally distributed with mean 0 and variance $t - s$. Hence (M_t) is an $(\mathcal{F}_t)$ Brownian motion. ♢

The vector case

In Chapter 3 we will need a vector version of the Lévy characterization. Thus, let $M_t = (M_t^1, \ldots, M_t^n)$ be an n-vector process each of whose components M_t^i is a continuous local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Note that for any two numbers a, b we have $ab = \frac{1}{4}((a+b)^2 - (a-b)^2)$. (This is sometimes called the 'polarization formula'.) With this in mind, define

$$[M^i, M^j]_t = \frac{1}{4}\left([M^i + M^j]_t - [M^i - M^j]_t \right).$$

The processes $[M^i + M^j], [M^i - M^j]$ are the quadratic variation processes of the local martingales $M^i + M^j$ and $M^i - M^j$ respectively, as introduced in Theorem 1.8 above. $[M^i, M^j]$ is sometimes called the 'cross-variation' of the local martingales M^i, M^j. Note that $[M^i, M^i] = [M^i]$. We can write the cross-variation as a symmetric $n \times n$ matrix, denoted $[M]_t$, with i, j'th component $[M^i, M^j]$. We leave it as an exercise for the reader to show that this matrix is almost surely non-negative definite. The vector version of the Ito formula (1.18) for continuous local semimartingales is as follows. We start with a vector of continuous local semimartingales $X_t^i = M_t^i + A_t^i$, $i = 1, \ldots, n$. Denote $[X^i, X^j] = [M^i, M^j]$. Then for any $C^{1,2}$ function $f : R^+ \times R^n \to R$, we have

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}dX^i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}d[X^i, X^j]. \tag{1.21}$$

A particularly useful special case of (1.21) is the *product formula*:

$$d(X_t^1 X_t^2) = X_t^1 dX_t^2 + X_t^1 dX_t^2 + d[X^1, X^2]_t. \tag{1.22}$$

Theorem 1.11. *Let M_t be an R^n-valued continuous local martingale as described above, and suppose that $[M]_t = \mathbf{I}t$, where $\mathbf{I}$ denotes the $n \times n$ identity matrix. Then the components $M_t^i, i = 1, \dots, n$ are independent Brownian motions.*

The proof of this result is essentially the same as that of the univariate case, Theorem 1.10 above. We show that under the condition stated, for any n-vector u and $t > s$,

$$E\left[e^{i<u, M_t - M_s>} \middle| \mathcal{F}_s \right] = e^{-\frac{1}{2}|u|^2(t-s)}.$$

As before, this shows that the increment $M_t - M_s$ is independent of $\mathcal{F}_s$ with distribution $N(0, \mathbf{I}(t-s))$. The conclusion follows.

1.4 The Girsanov Theorem

The Girsanov theorem states that, for Brownian motion, absolutely continuous change of measure is equivalent to change of drift.

Theorem 1.12. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space, where $0 < T < \infty$ and we assume for convenience that $\mathcal{F} = \mathcal{F}_T$. Let w_t be an $(\mathcal{F}_t, P)$-Brownian motion.*
(a) Let $g(t)$ be an adapted process satisfying $\int_0^T g^2(s)ds < \infty$ a.s. and define

$$\Lambda_T = \exp\left(\int_0^T g(s)dw_s - \frac{1}{2}\int_0^T g^2(s)ds \right). \tag{1.23}$$

Suppose that $E[\Lambda_T] = 1$, and define a measure Q on $(\Omega, \mathcal{F})$ by $dQ/dP = \Lambda_T$. Then under measure Q the process $\tilde{w}_t$ defined by

$$\tilde{w}_t = w_t - \int_0^t g(s)ds$$

is an $\mathcal{F}_t$ Brownian motion.
(b) Suppose $\mathcal{F}_t$ is the natural filtration of w_t and that Q is a measure such that $Q \sim P$. Then there exists a process $g(t)$ such that dQ/dP is equal to Λ_T defined by (1.23).

PROOF: (a) The assumption that $E\Lambda_T = 1$ ensures that Q is a probability measure. Applying the Ito formula, we find that

$$d(\tilde{w}\Lambda) = \Lambda(\tilde{w}g + 1)dw,$$

so that $\tilde{w}\Lambda$ is a local martingale which implies, as shown in section 1.2.3, that $\tilde{w}$ is a Q-local martingale. Certainly $\tilde{w}$ has continuous sample paths, and by the argument in section 1.3.1 the quadratic variation of $\tilde{w}$ is equal to t. By the Lévy characterization, $\tilde{w}$ is a Q-Brownian motion.
(b) Let Q be an equivalent measure and define $\Lambda_T = dQ/dP$. Then $\Lambda_T > 0$ a.s. and $E\Lambda_T = 1$. For any $t \in [0, T]$ let P^t, Q^t denote the restrictions of P and Q to $\mathcal{F}_t$. Then $P^t \sim Q^t$ and the Radon-Nikodym derivative is $dQ^t/dP^t := \Lambda_t = E[\Lambda_T|\mathcal{F}_t]$. Hence $\Lambda_t > 0$ a.s. By the martingale representation theorem for Brownian motion, there exists an integrand ϕ such that $\int_0^T \phi^2(t)dt < \infty$ and

$$\Lambda_t = 1 + \int_0^t \phi(s)dw_s, \quad 0 \le t \le T. \tag{1.24}$$

Now apply the Ito formula to calculate

$$d \log \Lambda_t = \frac{1}{\Lambda_t}\phi(t)dw_t - \frac{1}{2}\frac{1}{\Lambda_t^2}\phi^2(t)dt.$$

Thus Λ_T is given by (1.23) with $g(t) = \phi(t)/\Lambda_t$. ♢

Remarks (a) Let M_t be a non-negative local martingale, i.e. for times $\tau_n \uparrow \infty$, for $t > s$

$$M_{s\wedge\tau_n} = E[M_{t\wedge\tau_n}|\mathcal{F}_s].$$

Thus, using Fatou's lemma for conditional expectation,

$$\begin{aligned} M_s &= \liminf{}_n M_{s\wedge\tau_n} \\ &= \liminf{}_n E[M_{t\wedge\tau_n}|\mathcal{F}_s] \\ &\geq E[\liminf{}_n M_{t\wedge\tau_n}|\mathcal{F}_s] \\ &= E[M_t|\mathcal{F}_s]. \end{aligned}$$

Thus any non-negative local martingale is a supermartingale, so that in particular EM_t is a decreasing function of t. Now Λ_T defined by (1.23) is a non-negative local martingale, so the assumption that $E\Lambda_T = 1$ implies that $E\Lambda_t = 1$ for all $t \in [0, T]$, since $\Lambda_0 = 1$ a.s.

(b) The best general sufficient condition implying $E\Lambda_T = 1$ is the *Novikov condition*

$$E \exp\left(\frac{1}{2}\int_0^T g^2(s)ds\right) < \infty.$$

2

The Black Scholes World

2.1 The Model

To start with we consider a world with just one risky asset with price process S_t and a risk-free savings account paying constant interest rate r with continuous compounding. Everything takes place in a finite time interval $[0,T]$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, (w_t)_{t\in[0,T]})$ be Wiener space, i.e. w_t is Brownian motion, $\mathcal{F}_t$ is the natural filtration of w_t and $\mathcal{F} = \mathcal{F}_t$. The price process S_t is supposed to be *geometric Brownian motion*: S_t satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dw_t \tag{2.1}$$

for given *drift* μ and *volatility* σ. (2.1) has a unique solution: if S_t satifies (2.1) then by the Ito formula

$$d\log S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dw_t,$$

so that S_t satisfies (2.1) if and only if

$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma w_t). \tag{2.2}$$

Note that this makes S_t very easy to simulate: for any increasing sequence of times $0 = t_0 < t_1 \ldots$,

$$S_{t_i} = S_{t_{i-1}} \exp\left((\mu - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}X_i\right),$$

where $X_1, X_2, \ldots$ is a sequence of independent $N[0,1]$ random variables. This representation is *exact*. Another thing that follows from (2.2) is that $ES_t = S_0 e^{\mu t}$.

In the interests of symmetry we want the savings account also to be expressed as a traded asset, i.e. we should invest in it by buying a certain number of units of something. A convenient 'something' is a *zero-coupon bond*

$$B_t = \exp(-r(T-t)).$$

This grows, as required, at rate r:

$$dB_t = rB_t dt \tag{2.3}$$

Note that (2.3) does not depend on the final maturity T (the same growth rate is obtained from any ZC bond) and the choice of T is a matter of convenience as we will see below.

2.2 Portfolios and Trading Strategies

If we hold ϕ and ψ units of S and B respectively at time t then we have a portfolio whose time-t value is $\phi S_t + \psi B_t$. The assumptions of Black-Scholes are that we have a *frictionless market*, meaning that S and B can be traded in arbitrary amounts with no transaction costs, and short positions are allowed. In particular this means we can invest in, or borrow from, the riskless account at the same rate r of interest. A *trading strategy* is then a triple (ϕ_t, ψ_t, x_0), where x_0 is the initial endowment and (ϕ_t, ψ_t) is a pair of adapted processes satisfying

$$\int_0^T \phi_t^2 dt < \infty \quad \text{a.s.,} \qquad \int_0^T |\psi_t| dt < \infty.$$

The *gain from trade* in $[s, t]$ is then

$$\int_s^t \phi_u dS_u + \int_s^t \psi_u dB_u.$$

(Note how this matches up with the definition of the Ito integral!) A portfolio is *self-financing* if

$$\phi_t S_t + \psi_t B_t - \phi_s S_s - \psi_s B_s = \int_s^t \phi_u dS_u + \int_s^t \psi_u dB_u.$$

The increase in portfolio value is entirely due to gains from trade.

2.3 Arbitrage and Valuation

Denote by V_t the portfolio value at time t, i.e. $V_t = \phi_t S_t + \psi_t B_t$. An *arbitrage opportunity* is the existence of a self-financing trading strategy and a time t such that $V_0 = 0$, $V_t \geq 0$ a.s. and $P[V_t > 0] > 0$ (or, equivalently, $EV_t > 0$.) It is axiomatic that arbitrage cannot exist in the market, so no mathematical model should permit arbitrage opportunities.

2.3.1 Forwards

Consider a forward contract in which we fix a price K now to be paid at time T for delivery of 1 unit of S_T. *The unique no-arbitrage value of K is $F = e^{rT} S_0$.* Indeed, suppose someone offers us a forward contract at $K < F$. We sell one share and invest the proceeds S_0 in the bank. At time T we get the share back for a payment of K but the value of our bank account is $F > K$. We make a riskless profit of $F - K$. If we are able to offer a forward at $K > F$ then we should borrow S_0 and buy the share. Again, there is a riskless profit at time T. (This argument is independent of the pricing model for S_t.)

2.3.2 Put-Call Parity

A call option with strike K and exercise time T has exercise value $[S_T - K]^+$, and a put has exercise value $[K - S_T]^+$. Clearly

$$[S_T - K]^+ - [K - S_T]^+ = S_T - K,$$

so that buying a call and selling a put at time zero is equivalent to buying a forward and agreeing to pay K at time T. Thus whatever the prices C_0 and P_0 at time 0, they must satisfy

$$C_0 - P_0 = (F - K)B_0.$$

Note again that this is completely model-independent.

2.3.3 Replication

Suppose there is a contingent claim with exercise value $h(S_T)$ at time T (for example a put or call option) and there exists a self-financing trading strategy (ϕ, ψ, x_0) such that $V_T = h(S_T)$ a.s. Then x_0 is the unique no-arbitrage price of the contingent claim. The arbitrage, if available, is realized by selling the contingent claim and going long the replicating portfolio, or *vice versa*. This argument is sometimes known as the *law of one price*: if two assets have identical cash flows in the future then they must have the same value now.

2.4 Black-Scholes: the Original Proof

Black's and Scholes' original proof of the famous formula [1] was a very direct argument showing that a replicating portfolio exists for the European call option. Here is a version of that argument.

The idea is to assume a whole lot of things and then show they are all true. The first assumption is that there is a smooth function $C(t,S)$ such that the call option has a value $C(t,S_t)$ at time $t<T$, with $\lim_{t\uparrow T} C(t,S) = [S-K]^+$. Suppose we form a portfolio in which we are long one unit of the call option and short a self-financing portfolio $(\phi, \psi, C(0,S_0))$. The value of this portfolio at time t is then

$$X_t = C(t,S_t) - \phi_t S_t - \psi_t B_t,$$

with, in particular, $X_0 = 0$. By the Ito formula and the self-financing property,

$$dX_t = \frac{\partial C}{\partial S} dS + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt - \phi_t dS_t - \psi_t dB_t.$$

If we choose $\phi_t = \partial C / \partial S$ and use the fact that $dB = rBdt$ we see that

$$dX_t = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} dt - \psi_t r B_t \right) dt.$$

Let us now choose

$$\psi_t = \frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}}{rB_t},$$

heroically assuming that in doing so we have not destroyed that self-financing property. Then $X_t \equiv 0$, so that

$$C = \phi S + \psi B = S \frac{\partial C}{\partial S} + \frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}}{r},$$

showing that C must satisfy the Black-Scholes PDE

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{2.4}$$

with boundary condition

$$C(T,S) = [S-K]^+. \tag{2.5}$$

Equations (2.4),(2.5) are enough to determine the function C, as we will show below. Is (ϕ, ψ, x_0) in fact self-financing? By definition $\phi S + \psi B = C$ (since $X_t \equiv 0$) and

$$\begin{aligned} \int_0^t \phi dS + \int_0^t \psi dB &= \int_0^t \frac{\partial C}{\partial S} dS + \int_0^t \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) du \\ &= \int_0^t dC \\ &= C(t,S_t) - C(0,S_0). \end{aligned}$$

This confirms the self-financing property. We have now shown that the call option can be replicated by a self-financing portfolio with initial endowment $C(0, S_0)$, so by the argument in section 2.3.3 this is the unique arbitrage-free price.

In the above argument, no special role is played by the call option exercise function $[S_T - K]^+$. It simply provides the boundary condition for the Black-Scholes PDE (2.4). If we used another boundary condition $C(T, S) = h(S)$ then the corresponding solution of (2.4) would give us the no-arbitrage value and hedging strategy for a contingent claim with exercise value $h(S_T)$.

Example: the Forward Price. It is easy to check that

- $C(t, S) = S$ satisfies (2.4) with boundary condition $C(T, S) = S$.
- For a constant K, $C(t, S) = e^{-r(T-t)}K$ satisfies (2.4) with boundary condition $C(T, S) = K$.

Since (2.4) is a linear equation, the value at time 0 of receiving $S_T - K$ at time T is therefore $S_0 - Ke^{-rT}$, which is equal to zero when $K = e^{rT}S_0$, the forward price. You can check (please do!) that the hedging strategy $\phi = \partial C/\partial S$ implied by Black-Scholes coincides with the strategy given in section 2.3.1.

2.4.1 Probabilistic solution of the Black-Scholes PDE

Suppose we have an SDE

$$dx_t = m(x_t)dt + g(x_t)dw_t$$

where m, g are Lipschitz continuous functions so that a solution exists. The differential generator of x_t is the operator $\mathcal{A}$ defined by

$$\mathcal{A}f(x) = m(x)\frac{\partial f}{\partial x} + \frac{1}{2}g^2(x)\frac{\partial^2 f}{\partial x^2}$$

so the Ito formula can be written

$$df(x_t) = \mathcal{A}f(x_t)dt + \frac{\partial f}{\partial x}g(x_t)dw_t.$$

Now consider the following PDE for a function $v(t, x)$

$$\frac{\partial v}{\partial t} + \mathcal{A}v(t,x) - rv(t,x) = 0, \quad t < T, \tag{2.6}$$

$$v(T,x) = \Xi(x), \tag{2.7}$$

where r is a given constant and Ξ a given function. If v satisfies this then applying the Ito formula we find that

$$\begin{aligned} d(e^{-rt}v(t,x_t)) &= -re^{-rt}v(t,x_t)dt + e^{-rt}\left(\frac{\partial v}{\partial t} + \mathcal{A}v(t,x_t)dt + \frac{\partial v}{\partial x}g(t,x_t)dw_t\right) \\ &= e^{-rt}\frac{\partial v}{\partial x}g(t,x_t)dw_t. \end{aligned} \tag{2.8}$$

Thus $\exp(-rt)v(t, x_t)$ is a local martingale. If it is a martingale then integrating from t to T and using (2.7) we see that

$$v(t,x_t) = E_{t,x}\left[e^{-r(T-t)}\Xi(x_T)\right]$$

This is the probabilistic representation of the solution of the PDE (2.6),(2.7).

Comparing (2.4),(2.5) with (2.6),(2.7) we see that these equations match up when $m(x) = rx$ and $g(x) = \sigma x$, i.e. x_t satisfies

$$dx_t = rx_t\,dt + \sigma x_t\,dw_t. \tag{2.9}$$

Now return to the price model (2.1) and introduce a measure change

$$\frac{dQ}{dP} = \exp\left(\alpha w_T - \frac{1}{2}\alpha^2 T\right),$$

where α is a constant. By Girsanov, $d\check{w} = dw - \alpha dt$ is a Q-Brownian motion, in terms of which (2.1) becomes

$$dS_t = \mu S_t\, dt + \sigma S_t(d\check{w}_t + \alpha\, dt).$$

Choosing $\alpha = (r - \mu)/\sigma$ we get

$$dS_t = rS_t\, dt + \sigma S_t d\check{w}_t, \tag{2.10}$$

the same equation as (2.9). Equation (2.10) is the price process expressed in the *risk-neutral measure* Q, and the above argument shows that the probabilistic solution of the Black-Scholes PDE (2.4),(2.5) is

$$C(t, S) = E^Q_{t,S}\left(e^{-r(T-t)}[S_T - K]^+\right).$$

This is however easily computed since S_T is given explicitly in terms of w_T by (2.2) (with r replacing μ). We get

$$C(t, S) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [S \exp((r - \sigma^2/2)(T - t) - \sigma x\sqrt{T - t}) - K]^+ e^{-x^2/2} dx.$$

A short calculation gives the final expression

$$C(t, S) = SN(d_1) - e^{-r(T-t)}KN(d_2) \tag{2.11}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function and

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = d_1 - \sigma\sqrt{T - t}$$

We can now tie up the loose ends of the argument. It can be checked directly that C defined by (2.11) does satisfy the Black-Scholes PDE (2.4),(2.5), and another short calculation (see Problems II!) shows that $\partial C/\partial S = N(d_1)$, so that in particular $0 < \partial C/\partial S < 1$. Hence the integrand in (2.8) is square-integrable and the stochastic integral is a martingale, as required.

Another version of the formula, often more useful, is this. Recall that the *forward price* at t for delivery at T is $F = Se^{r(T-t)}$. We can therefore express (2.11) as

$$C(t, S) = e^{-r(T-t)}(FN(d_1) - KN(d_2)), \tag{2.12}$$

and d_1 can be expressed as

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T - t)}{\sigma\sqrt{T - t}}.$$

2.5 Proof by Martingale Representation

Let ϕ be an adapted process with

$$\int_0^T \phi_t^2 S_t^2\, dt < \infty \text{ a.s.} \tag{2.13}$$

and let X_t be a process defined by

$$dX_t = \phi_t\, dS_t + (X_t - \phi_t S_t)r\, dt, \quad X_0 = x_0. \tag{2.14}$$

The interpretation is that X_t is the portfolio value corresponding to the trading strategy (ϕ, ψ, x_0) where

$$\psi_t = \frac{X_t - \phi_t S_t}{B(t)}, \tag{2.15}$$

i.e. ϕ_t units of the risky asset are held and the remaining value $X_t - \phi_t S_t$ is held in the savings account. This strategy is always self-financing since X_t is by definition the gains from trade process, while the value is $\phi S + \psi B = X$. Applying the Ito formula we find that, with $\tilde{X}_t = e^{-rt} X_t$,

$$d\tilde{X}_t = \phi e^{-rt} S((\mu - r)\, dt + \sigma\, dw) \tag{2.16}$$

$$= \phi \tilde{S} \sigma d\check{w}, \tag{2.17}$$

where $\tilde{S}_t = e^{-rt} S_t$. (The first line (2.16) shows incidentally that (2.14) has a unique solution.) Thus $e^{-rt} X_t$ is a local martingale in the risk-neutral measure Q.

Suppose we have an option whose exercise value at time T is H, where H is an $\mathcal{F}_T$-measurable random variable with $EH^2 < \infty$. By the martingale representation theorem there is an integrand g such that

$$e^{-rT} H = E_Q[e^{-rT} H] + \int_0^T g_t\, d\check{w}_t.$$

Define $\phi_t = e^{rt} g_t / (\sigma S_t)$ and then ψ by (2.15) and $x_0 = E_Q[e^{-rT} H]$. Then $\phi \tilde{S} \sigma = g$ and the trading strategy (ϕ, ψ, x_0) generates a portfolio value process such that $X_T = H$ a.s., i.e. (ϕ, ψ, x_0) is a replicating portfolio for H. It follows that the option value is

$$x_0 = E_Q[e^{-rT} H].$$

Note this is a much more general result than that obtained by the previous argument, in that the option payoff can be an arbitrary, possibly ‘path-dependent’, random variable, whereas before we assumed it took a value of the form $H = h(S_T)$. On the other hand the above argument only asserts that a replicating portfolio exists: it does not give an explicit formula for ϕ.

Theorem 2.1. *Let Φ be the class of investment strategies ϕ_t such that (a) the integrability condition (2.13) is satisfied, and (b) there exists a positive constant A_ϕ such that $X_t \geq -A_\phi$ for all $t \in [0, T]$, where X_t is the process defined by (2.14). In the Black-Scholes model, no strategy $\phi \in \Phi$ is an arbitrage opportunity.*

PROOF: Suppose X_t is given by (2.14) for some strategy $\phi \in \Phi$ and $X_0 = 0$. Then, from (2.17), the discounted process $\tilde{X}_t$ is a Q-local martingale which is bounded below by the constant $-A_\phi$. Thus $\tilde{X}_t + A_\phi$ is a non-negative local martingale, and hence a supermartingale. Therefore $\tilde{X}_t$ is a supermartingale and has decreasing expectation: for any $t > 0$

$$0 = E_Q[\tilde{X}_0] \geq E_Q[\tilde{X}_t]. \tag{2.18}$$

On the other hand, if $X_t \geq 0$ a.s.(P) and $P[X_t > 0] > 0$ then, since P and Q are equivalent measures, $E_Q[\tilde{X}_t] > 0$, which is incompatible with (2.18). Hence there cannot be an arbitrage opportunity as defined in Section 2.3. ◇

2.6 Robustness of Black-Scholes Hedging

If we assume the Black-Scholes price model (2.1) then the price at time t of an option with exercise value $h(S_T)$ is $C_h(S_t, r, \sigma, t) = C(t, S_t)$ where $C(t, S)$ satisfies the Black-Scholes PDE (2.4) with boundary condition $C(T, S) = h(S)$.

Suppose we sell an option at implied volatility $\hat{\sigma}$, i.e. we receive at time 0 the premium $C_h(S_0, r, \hat{\sigma}, 0)$, and we hedge under the assumption that the model (2.1) is correct with $\sigma = \hat{\sigma}$. The hedging strategy is then 'delta hedging': the number of units of the risky asset held at time t is the so-called option 'delta' $\partial C/\partial S$:

$$\phi_t = \frac{\partial C}{\partial S}(t, S_t). \tag{2.19}$$

Suppose now that the model (2.1) is *not* correct, but the 'true' price model is

$$dS_t = \alpha(t,\omega)S_t dt + \beta(t,\omega)S_t dw_t, \tag{2.20}$$

where w_t is an $\mathcal{F}_t$-Brownian motion for some filtration $\mathcal{F}_t$ (not necessarily the natural filtration of w_t) and α_t, β_t are $\mathcal{F}_t$-adapted, say bounded, processes. It is no loss of generality to write the drift and diffusion in (2.20) as $\alpha S, \beta S$: since $S_t > 0$ a.s. we could always write a general diffusion coefficient γ as $\gamma_t = (\gamma_t/S_t)S_t \equiv \alpha_t S_t$. In fact the model (2.20) is saying little more than that S_t is a positive process with continuous sample paths.

Using strategy (2.19) the value X_t of the hedging portfolio is given by $X_0 = C(0, S_0)$ and

$$dX_t = \frac{\partial C}{\partial S} dS_t + \left(X_t - \frac{\partial C}{\partial S} S_t\right) r\, dt$$

where S_t satisfies (2.20). By the Ito formula, $Y_t \equiv C(t, S_t)$ satisfies

$$dY_t = \frac{\partial C}{\partial S} dS + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial S^2}\right) dt.$$

Thus the hedging error $Z_t \equiv X_t - Y_t$ satisfies

$$\frac{d}{dt} Z_t = rX_t - rS_t \frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial S^2}.$$

Using (2.4) and denoting $\Gamma_t = \Gamma(t, S_t) = \partial^2 C(t, S_t)/\partial s^2$, we find that

$$\frac{d}{dt} Z_t = rZ_t + \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2).$$

Since $Z_0 = 0$, the final hedging error is

$$\boxed{Z_T = X_T - h(S_T) = \int_0^T e^{r(T-s)} \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2) dt.}$$

Comments:

This is a key formula, as it shows that successful hedging is quite possible even under significant model error. It is hard to imagine that the derivatives industry could exist at all without some result of this kind. Notice that:

- Successful hedging depends entirely on the relationship between the Black-Scholes implied volatility $\hat{\sigma}$ and the true 'local volatility' β_t. For example, if we are lucky and $\hat{\sigma}^2 \geq \beta_t^2$ a.s. for all t then the hedging strategy (2.19) makes a profit *with probability one* even though the true price model is substantially different from the assumed model (2.1), as long as $\Gamma_t \geq 0$, which holds for standard puts and calls.
- The hedging error also depends on the option convexity Γ. If Γ is small then hedging error is small even if the volatility has been underestimated.

2.7 Options on Dividend-paying Assets

Holders of ordinary shares receive dividends, which are cash payments normally quoted as "x pence per share", paid on specific dates with the value x being announced some time in advance. For a stock index, where the constituent stocks are all paying different dividends at different times, it makes sense to think in terms of a *dividend yield*, the dividend per unit time expressed as a fraction of the index value. In mathematical terms, we assume that a dividend is a continuous-time payment stream, the dividend paid in a time interval dt being $qS_t dt$. Thus q is the dividend yield. In this section we analyse the case where q is a fixed constant. Equation (2.14), describing the evolution of a self-financing portfolio, must be modified to

$$dX_t = \phi_t\, dS_t + q\phi_t S_t dt + (X_t - \phi_t S_t) r\, dt, \quad X_0 = x_0. \tag{2.21}$$

$$= \phi_t S_t(\mu + q - r)dt + X_t r\, dt + \phi_t S_t \sigma dw_t, \tag{2.22}$$

so that

$$d\left(e^{-rt} X_t\right) = \phi\tilde{S}(\mu + q - r)dt + \phi\tilde{S}\sigma\, dw. \tag{2.23}$$

Now change to a martingale measure Q_q such that

$$dw^q = dw + \frac{\mu + q - r}{\sigma} dt$$

is a Q_q-Brownian motion. Then (2.23) becomes simply

$$d\left(e^{-rt} X_t\right) = \phi\tilde{S}\sigma\, dw^q.$$

Thus by the argument of the previous section, the price at time 0 of a contingent claim H is

$$p = E_{Q^q}\left[e^{-rT} H\right]. \tag{2.24}$$

In particular, take $H = S_T$. Then p is the no-arbitrage price now for delivery of 1 unit of the asset at time T, or, equivalently, $e^{rT}p$ is the forward price.

In (2.21), take $X - \phi S = 0$, so that all receipts are re-invested in the risky asset S, nothing being held in the bank account. Then $\phi = X/S$, so that

$$\begin{aligned} dX &= \frac{X}{S} dS + q\frac{X}{S} S\, dt \\ &= X(\mu\, dt + \sigma\, dw) + qX\, dt \\ &= X((\mu + q)dt + \sigma\, dw). \end{aligned} \tag{2.25}$$

On the other hand, if we define $\hat{S}_t = e^{qt} S_t$ and use (2.1) and the Ito formula, we find that

$$d\hat{S}_t = \hat{S}_t((\mu + q)dt + \sigma\, dw). \tag{2.26}$$

From (2.25) and (2.26) we see that $X_t = \hat{S}_t = e^{qt} S_t$ for all $t > 0$ if $X_0 = S_0$. Now the solution of (2.25) is linear in the initial condition, so if $X_0 = e^{-qT} S_0$ then $X_T = S_T$ a.s. We have shown the following.

Proposition 2.2. *(i) For an asset with a constant dividend yield q, the forward price at time T is $F_T = e^{(r-q)T} S_0$. The replicating strategy that delivers one unit of the asset at time T consists of buying e^{-qT} units of the asset at time 0 and reinvesting all dividends in the asset.*

(ii) The value of a call option on the asset with exercise time T and strike K is

$$C(S_0, K, r, q, \sigma, T) = e^{-rT}(F_T N(d_1) - K N(d_2)), \tag{2.27}$$

where

$$d_1 = \frac{\log(F_T/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Proof: Only part (ii) remains to be proved. Under measure Q_q, S_t satisfies

$$dS_t = (r-q)S_t dt + \sigma S_t dw_t^q, \tag{2.28}$$

and the option value is given by (2.24) with $H = [S_T - K]^+$. This is exactly the same calculation as standard Black-Scholes, but with $(r-q)$ replacing r in the price equation (2.28) (but not in the 'discount factor' e^{-rT} in (2.24)). Formula (2.27) follows from (2.12). ♢

2.8 Barrier Options

Let S_t be a price process and let $M_t = \max_{0\le u\le t} S_u$ be the maximum price to date. An *up-and-out* call option has exercise value

$$[S_T - K]^+ \mathbf{1}_{M_T<B}.$$

It pays the standard call payoff if $S_t < B$ for all $t \in [0,T]$ and zero otherwise. B is the 'barrier', and to make sense, we must have $S_0 < B, K < B$. An *up-and-in* call option pays

$$[S_T - K]^+ \mathbf{1}_{M_T\ge B}.$$

The sum of these two payoffs is an ordinary call, so we only need to value one of the above. There are analogous definitions for down-and-out and down-and-in options.

Remarkably, there are analytic formulas for the values of these options in the Black-Scholes world. These formulas – but not the proofs – can be found on pages 462-464 of Hull's book [5]

The starting point is the so-called reflection principle for Brownian motion. Let x_t be standard Brownian motion starting at zero and $m_t = \max_{s\le t} x_s$. The reflection principle states that for $y>0$ and $x \le y$,

$$P[m_t \ge y, x_t < x] = N\left(\frac{x-2y}{\sqrt{t}}\right). \tag{2.29}$$

The idea is that those paths that do hit level y before time t 'restart' from level y with symmetric distribution (see figure 2.1), so there is equal probability that they will be below $x = y-(y-x)$ or above $y+(y-x) = 2y-x$ at time t. But

$$\begin{aligned} P[m_t \ge y, x_t \ge 2y-x] &= P[x_t \ge 2y-x] \\ &= 1 - N\left(\frac{2y-x}{\sqrt{t}}\right) \\ &= N\left(\frac{x-2y}{\sqrt{t}}\right) \end{aligned}$$

Now $[x_t < x] = [m_t < y, x_t < x] \cup [m_t \ge y, x_t < x]$, so

$$N\left(\frac{x}{\sqrt{t}}\right) = P[x_t < x] = P[m_t < y, x_t < x] + P[m_t \ge y, x_t < x]. \tag{2.30}$$

We have shown the following.

Proposition 2.3. *The joint distribution of x_t, the Brownian motion at time t, and its maximum-to-date m_t is given by*

$$F_0(y,x) = P[m_t < y, x_t < x] = N\left(\frac{x}{\sqrt{t}}\right) - N\left(\frac{x-2y}{\sqrt{t}}\right) \tag{2.31}$$

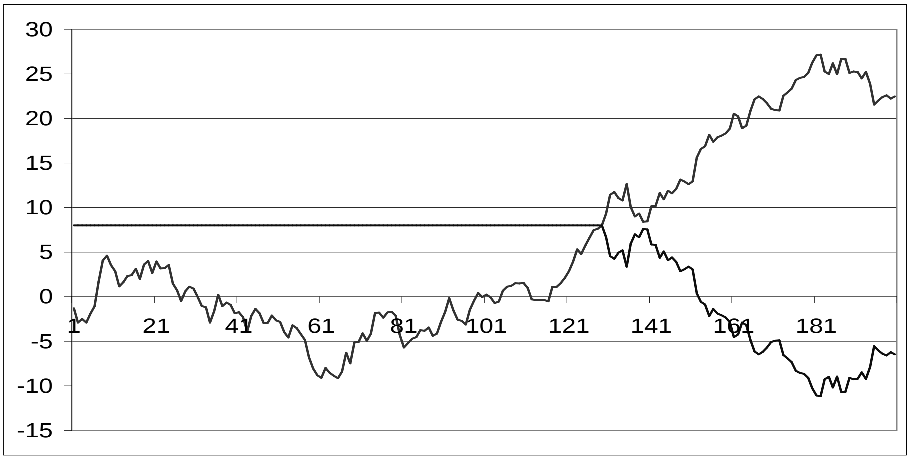


Fig. 2.1. Reflection principle

This argument depends on symmetry and doesn't work if x_t has drift. We can use the Girsanov theorem to get the answer in this case. If P_ν denotes the probability measure of BM with drift ν (i.e. $x_t = \nu t + w_t$ where w_t is ordinary BM) then we know that on the interval $[0, T]$

$$\frac{dP_\nu}{dP_0} = \exp(\nu x_T - \frac{1}{2}\nu^2 T).$$

Thus if f is any integrable function then, using (2.31),

$$\begin{aligned} E_\nu[\mathbf{1}_{m_T<y} f(x_T)] &= E_0\left[\mathbf{1}_{m_T<y} f(x_T)\frac{dP_\nu}{dP_0}\right] \\ &= E_0\left[\mathbf{1}_{m_T<y} f(x_T)\exp\left(\nu x_T - \frac{1}{2}\nu^2 T\right)\right] \\ &= \int_{-\infty}^{y} f(x)e^{(\nu x-\nu^2 T/2)}\frac{1}{\sqrt{T}}\left(\phi(x/\sqrt{T}) - \phi((x-2y)/\sqrt{T})\right) dx, \end{aligned}$$

where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard normal density function. Now clearly

$$\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{1}{2T}x^2 + \nu x - \frac{1}{2}\nu^2 T\right) = \frac{1}{\sqrt{T}}\phi\left(\frac{x-\nu T}{\sqrt{T}}\right)$$

while after some calculation we find that

$$\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{1}{2T}(x-2y)^2 + \nu x - \frac{1}{2}\nu^2 T\right) = \frac{e^{2y\nu}}{\sqrt{T}}\phi\left(\frac{x-2y-\nu T}{\sqrt{T}}\right).$$

This gives us the final result: the joint distribution function with drift ν is

$$F_\nu(y,x) = P_\nu[m_T < y, x_T < x] = \left(N\left(\frac{x-\nu T}{\sqrt{T}}\right) - e^{2y\nu}N\left(\frac{x-2y-\nu T}{\sqrt{T}}\right)\right). \tag{2.32}$$

This does coincide with F_0 when $\nu = 0$. A good reference for the above argument is Harrison [3].

Let us now return to barrier option pricing. The price process in the risk-neutral measure is

$$S_T = S_0 \exp((r - \sigma^2/2)T + \sigma w_T)$$

which we can write as

$$S_T = S_0 e^{\sigma x_T}$$

where $x_T = w_T + \nu T$ with

$$\nu = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right).$$

The price S_T is in the money but below the barrier level when $x_T \in (a_1, a_2)$ where

$$a_1 = \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right), \qquad a_2 = \frac{1}{\sigma}\log\left(\frac{B}{S_0}\right).$$

Denoting $g(y,x) = \partial F_\nu(y,x)/\partial x$, the option value can now be expressed as

$$E_\nu\left[e^{-rT}[S_T - K]^+ \mathbf{1}_{M_T<B}\right] = e^{-rT}\int_{a_1}^{a_2}(S_0 e^{\sigma x} - K)g(y,x)dx.$$

Doing the calculations we obtain the option value given in [5] as a sum of four terms of the form $c_1 N(c_2)$, as in the Black-Scholes formula. The up-and-out option price is

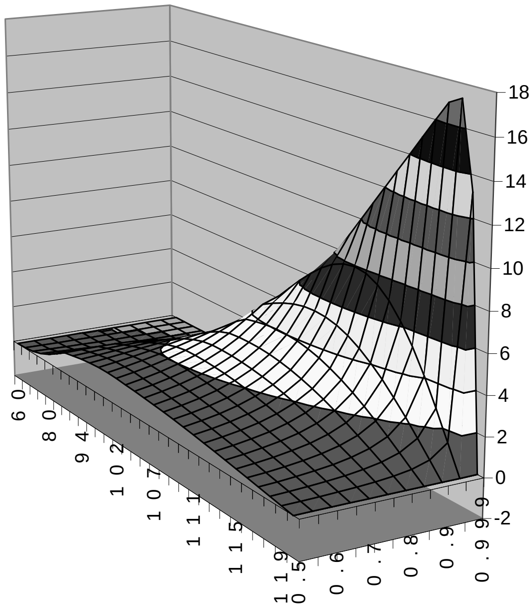
$$S_0\left(N(d_1) - N(x_1) + \left(\frac{B}{S_0}\right)^{2\lambda}(N(-y) - N(-y_1))\right)$$

$$+Ke^{-rT}\left(-N(d_2) + N(x_1 - \sigma\sqrt{T}) - \left(\frac{B}{S_0}\right)^{2\lambda-2}(N(-y+\sigma\sqrt{T}) - N(-y_1+\sigma\sqrt{T})\right)$$

where d_1, d_2 are the usual coefficients and

$$\begin{aligned} x_1 &= \frac{\log(S_0/B)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\ y_1 &= \frac{\log(B/S_0)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\ y &= \frac{\log(B^2/(S_0K))}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\ \lambda &= \frac{r + \sigma^2/2}{\sigma^2} \end{aligned}$$

Figures 2.2,2.3,2.4 show the value, delta and gamma of an up-and-out call option with strike $K = 100$, barrier level $B = 120$ and volatility 25%. The option matures at time $T = 1$. One can clearly see the "black hole" of barrier options: the region where the time-to-go is short and the priced is close to the barrier. In this region there is high negative delta, and there comes a point where hedging is essentially impossible because of the large gamma (i.e. unrealistically frequent rehedging is called for by the theory.)



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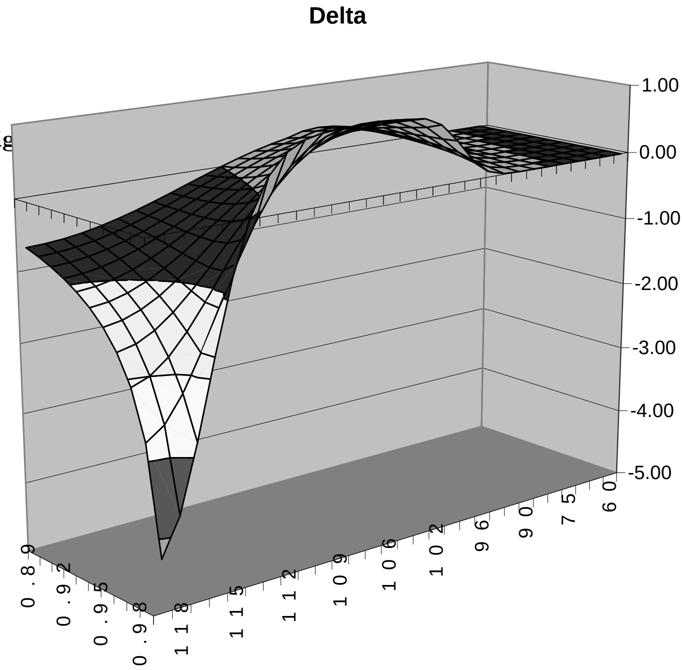

Fig. 2.3. Barrier option delta

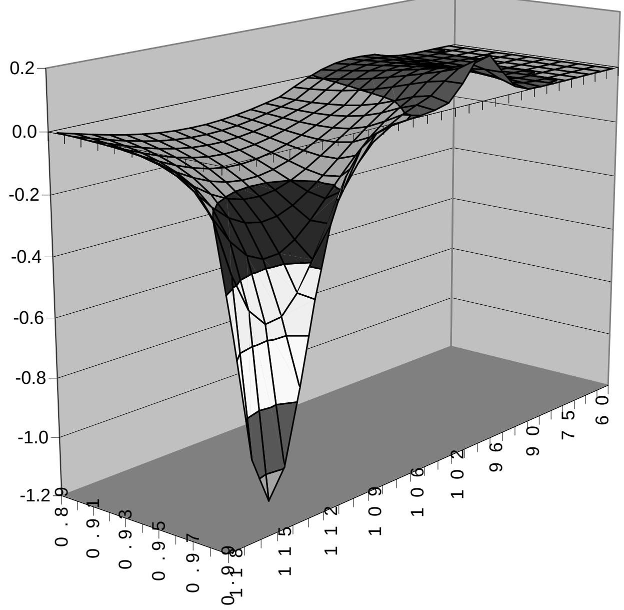

Fig. 2.4. Barrier option gamma

3

Multi-Asset Options

This chapter covers pricing of options where the exercise value depends on more than one risky asset. Section 3.1 describes a very useful formula for pricing exchange options, while Section 3.2 gives a model for the FX market, where the option could be directly an FX option or an option on an asset denominated in a foreign currency. Finally, in Section 3.3 we introduce the ideas of numéraire assets and changes of numéraire. These ideas give extra insight into the exchange option (and simpler calculations) and play a big role in interest rate theory.

3.1 The Margrabe Formula

This is an expression, originally derived by Margrabe [6], for the value

$$C(t_0, s_1, s_2) = E[e^{-r(T-t_0)} \max(S_1(T) - S_2(T), 0)] \tag{3.1}$$

of the option to exchange asset 2 for asset 1 at time T. It is assumed that under the risk-neutral measure P, $S_1(t)$ and $S_2(t)$ satisfy

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dw_1, \quad S_1(t_0) = s_1 \tag{3.2}$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dw_2, \quad S_2(t_0) = s_2, \tag{3.3}$$

where w_1, w_2 are Brownian motions with $E[dw_1 dw_2] = \rho dt$. The riskless rate is r. The Margrabe formula is

$$C(t_0, s_1, s_2) = s_1 N(d_1) - s_2 N(d_2) \tag{3.4}$$

where $N(\cdot)$ is the normal distribution function,

$$d_1 = \frac{\ln(s_1/s_2) + \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}} \tag{3.5}$$

$$d_2 = d_1 - \sigma\sqrt{T-t_0} \tag{3.6}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{3.7}$$

The following important facts can be noted right away.

1. The solutions of (3.2), (3.3) are

$$S_i(t) = s_i e^{r(t-t_0)} M_i(t_0, t), \quad i = 1, 2, \tag{3.8}$$

where

$$M_i(t_0, t) = \exp\left(\sigma_i(w_i(t) - w_i(t_0)) - \frac{1}{2}\sigma_i^2(t-t_0)\right), \quad i = 1, 2. \tag{3.9}$$

The process $t \mapsto M_i(t_0, t)$ is a martingale, and $M_i(t_0, t_0) = 1$, so in particular $EM_i(t_0, t) = 1$.

2. The function C defined by (3.1) does not depend on the riskless rate r. Indeed, from (3.8) we see that $e^{r(T-t_0)}$ is a factor of $S_1(T)$ and $S_2(T)$, and this cancels the discount factor $e^{-r(T-t_0)}$ in (3.1).
3. The function C is *homogeneous of degree 1*, i.e. for any $\lambda > 0$ we have

$$C(t_0, \lambda s_1, \lambda s_2) = \lambda C(t_0, s_1, s_2). \tag{3.10}$$

This is evident from the definition of the exercise value in (3.1) and the fact—seen from (3.8)—that $S_i(T)$ is linear in the initial condition s_i.
Suppose that the function C is continuously differentiable in s_1 and s_2. Then differentiating both sides of (3.10) with respect to λ and setting $\lambda = 1$ we obtain the key relationship

$$s_1 \frac{\partial C}{\partial s_1} + s_2 \frac{\partial C}{\partial s_2} = C \tag{3.11}$$

3.1.1 The Probabilistic Method

Since C does not depend on the riskless rate r, we may and shall assume that $r = 0$. Without loss of generality, we also take $t_0 = 0$. Then

$$\begin{aligned} C &= E[\max(S_1(T) - S_2(T), 0)] \\ &= E\left[S_2(T) \max\left(\frac{S_1(T)}{S_2(T)} - 1, 0\right)\right] \end{aligned} \tag{3.12}$$

By the Ito formula, $Y(t) = S_1(t)/S_2(t)$ satisfies

$$dY = Y(\sigma_2^2 - \sigma_1\sigma_2\rho)dt + Y(\sigma_1 dw_1 - \sigma_2 dw_2). \tag{3.13}$$

Now $S_2(t) = s_2 M_2(0,t)$ and we can regard $M_2(0,T)$ as a Girsanov exponential defining a measure change

$$\frac{d\tilde{P}}{dP} = M_2(T). \tag{3.14}$$

Thus from (3.12)

$$C = s_2 \tilde{E}[\max(Y(T) - 1, 0)] \tag{3.15}$$

where $\tilde{E}$ denotes expectation under measure $\tilde{P}$. By the Girsanov theorem, under measure $\tilde{P}$ the process

$$d\tilde{w}_2 = dw_2 - \sigma_2 dt$$

is a Brownian motion. We can write w_1 as $w_1(t) = \rho w_2(t) + \sqrt{1-\rho^2}\, w'(t)$ where $w'(t)$ is a Brownian motion independent of $w_2(t)$ (under measure P). It is shown below in Lemma 3.1 that w' *remains a Brownian motion under* $\tilde{P}$*, independent of* $\tilde{w}_2$. Hence $d\tilde{w}_1$ defined by

$$\begin{aligned} d\tilde{w}_1 &= \rho d\tilde{w}_2(t) + \sqrt{1-\rho^2}\, dw'(t) \\ &= dw_1(t) - \rho\sigma_2 dt \end{aligned}$$

is a $\tilde{P}$-Brownian motion. Using (3.13), we find that the equation for Y under $\tilde{P}$ is

$$dY = Y(\sigma_1 d\tilde{w}_1 - \sigma_2 d\tilde{w}_2)$$

which we can write

$$dY = Y\sigma dw, \tag{3.16}$$

where w is a standard Brownian motion and σ is given by (3.7). We will see later in Section 3.3 just why it is that Y is a $\tilde{P}$-martingale. In view of (3.15), (3.16) the exchange option is equivalent to a call option on asset Y with volatility σ, strike 1 and riskless rate 0. By the Black-Scholes formula, this is (3.4).

REMARK: We can recover Black-Scholes from Margrabe simply by taking $\sigma_2 = 0$ and $s_2 = e^{-r(T-t_0)}K$; then $S_2(T) = K$ a.s.

Lemma 3.1. *Suppose B_t, B'_t are independent Brownian motions on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, P)$, and that ϕ_t is an adapted process such that $\int_0^T \phi_s^2 ds < \infty$ and $E\Lambda(T) = 1$, where*

$$\Lambda(t) = \exp\left(\int_0^t \phi_s dB_s - \frac{1}{2}\int_0^t \phi_s^2 ds\right).$$

Define a measure $\tilde{P}$ on $(\Omega, \mathcal{F}_T)$ by taking $d\tilde{P}/dP = \Lambda(T)$, and a process $\tilde{B}$ by $d\tilde{B}_t = dB_t - \phi_t dt$. Then $\tilde{B}_t, B'_t$ are independent Brownian motions under measure $\tilde{P}$.

Proof. This uses the Girsanov theorem, Theorem 1.12, together with the vector version, Theorem 1.11, of the Lévy characterization theorem. From the Girsanov theorem we know that $\tilde{B}_t$ is a $\tilde{P}$-Brownian motion. Since B, B' are independent under P we find by applying the Ito product formula (1.22) that $\Lambda B'$ is a P-local martingale and hence B' is a $\tilde{P}$-local martingale. Its quadratic variation is t so by the Lévy characterization B' is a $\tilde{P}$-Brownian motion. To complete the proof we have to show that $\tilde{B}, B'$ are independent under $\tilde{P}$. Under P, the process $B + B'$ has quadratic variation $2t$. Since $B + B'$ and $\tilde{B} + B'$ differ by a process of bounded variation, this shows that $[\tilde{B} + B'] = 2t$. By the same argument, $[\tilde{B} - B'] = 2t$, and hence $[\tilde{B}, B'] = 0$. We have thus shown that the vector process $M_t = (\tilde{B}_t, B'_t)$ is a $\tilde{P}$ local martingale with cross-variation process $[M]_t = \mathbf{I}t$. Hence $\tilde{B}_t, B'_t$ are independent Brownian motions under $\tilde{P}$, by Theorem 1.11.

3.1.2 Hedging an Exchange Option

Having determined the value of the exchange option, we now want to find the hedging strategy that replicates its exercise value. Note that there are in principle three traded assets: S_1, S_2 and the zero-coupon bond $P(t, T)$. In fact, the Margrabe hedging strategy only invests in two of them: S_1 and S_2. We get some hint of this from the fact that the option value does not depend on r; it is hard to imagine how this could be the case if hedging were to involve the riskless asset. The key to this question is the homogeneity property and specifically the property (3.10) of the Margrabe value C.

Recall that a trading strategy is a triple of processes $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$ whose values at time t are the number of units of S_1, S_2 and P, respectively, held in the hedging portfolio at time t. With C equal to the Margrabe value (3.4), define

$$\alpha_1(t) = \frac{\partial C}{\partial s_1}(t, S_1(t), S_2(t)), \quad \alpha_2(t) = \frac{\partial C}{\partial s_2}(t, S_1(t), S_2(t)), \quad \alpha_3(t) = 0. \tag{3.17}$$

Then (3.10) states that $C(t, S_1(t), S_2(t)) = \alpha_1(t)S_1(t) + \alpha_2(t)S_2(t)$, showing that this strategy is automatically replicating since in particular $C(T, S_1(T), S_2(T))$ coincides with the Margrabe exercise value. We only need to show that this strategy is self-financing which, in view of (3.17), is equivalent to showing that

$$dC = \alpha_1 dS_1 + \alpha_2 dS_2.$$

However, applying the Ito formula, we have

$$dC = \alpha_1 dS_1 + \alpha_2 dS_2 + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 C}{\partial s_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 C}{\partial s_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial s_1 \partial s_2}$$

so a sufficient condition for the self-financing property is that C satisfies the PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_1^2 s_1^2 \frac{\partial^2 C}{\partial s_1^2} + \frac{1}{2}\sigma_2^2 s_2^2 \frac{\partial^2 C}{\partial s_2^2} + \rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 C}{\partial s_1 \partial s_2} = 0. \tag{3.18}$$

Taking $\lambda = 1/s_2$ in (3.10) we have

$$C(t, s_1, s_2) = s_2 f(t, s_1/s_2) \tag{3.19}$$

where $f(t, y) = C(t, y, 1)$. We can now calculate derivatives of C in terms of those of f; for example

$$C_1 = \frac{\partial f}{\partial y}, \quad C_2 = f - \frac{s_1}{s_2}\frac{\partial f}{\partial y}.$$

Substituting these expressions into (3.18) we find that (3.18) is equivalent to the following PDE for f:

$$\frac{\partial f}{\partial t} + \frac{1}{2}y^2\sigma^2\frac{\partial^2 f}{\partial y^2} = 0, \tag{3.20}$$

where σ is as defined above. The boundary condition is

$$f(T, y) = C(T, y, 1) = \max(y - 1, 0). \tag{3.21}$$

But (3.20) (3.21) is just the Black-Scholes PDE whose solution is (3.4). Thus the strategy given by (3.17) is self-financing and replicates the Margrabe option exercise value.

3.1.3 Exercise Probability

As in Section 3.1.1, define $Y(t) = S_1(t)/S_2(t)$. We see from (3.12) that exercise takes place when $Y(T) > 1$, and under the risk-neutral measure $Y(t)$ satisfies (3.13). Hence the forward is $F = (s_1/s_0)\exp((\sigma_2^2 - \sigma_1\sigma_2\rho)T)$, and by standard calculations

$$\text{Risk-neutral probability of exercise} = P[Y(T) > 1] = N(\hat{d}_2),$$

where

$$\begin{aligned}\hat{d}_2 &= \frac{\ln(F) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ &= \frac{\ln(s_1/s_2) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)T}{\sigma\sqrt{T}}.\end{aligned}$$

Note that the exercise probability is *not* $N(d_2)$, which is the exercise probability under the transformed measure $\tilde{P}$.

3.1.4 Margrabe with dividends

If the assets S_1, S_2 pay constant dividend yields q_1, q_2 respectively, then the Margrabe formula becomes

$$C = s_1 e^{-q_1(T-t_0)}N(d_1) - s_2 e^{-q_2(T-t_0)}N(d_2),$$

where d_1, d_2 are given as before by (3.5), (3.6).

We leave it to the reader to derive this result, using the approach of Section 2.7.

3.1.5 Black-Scholes as a special case

Finally, let us see how we recover Black-Scholes as a special case. Indeed, if we set $\sigma_2 = 0$ and $s_2 = Ke^{-r(T-t_0)}$ then $S_2(T) = K$ a.s. and the option exercise value is value of a call option on S_1 with strike K. Thus if $D(t_0, s)$ denotes the value of this option then $D(t_0, s) = C(t_0, s, e^{-r(T-t_0)}K)$. The PDE (3.18) reduces to

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_1^2 s_1^2 \frac{\partial^2 C}{\partial s_1^2} = 0. \tag{3.22}$$

Now

$$\frac{\partial^2 D}{\partial s^2}(t,s) = \frac{\partial^2 C}{\partial s_1^2}(t, s, e^{-r(T-t_0)}K)$$

and

$$\frac{\partial D}{\partial t} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial s_2} r e^{-r(T-t)} K = \frac{\partial C}{\partial t} + r\left(C - s\frac{\partial C}{\partial s_1}\right),$$

where the second equality uses (3.10). Substituting in (3.22), we find that D satisfies

$$\frac{\partial D}{\partial t} + rs\frac{\partial D}{\partial s} + \frac{1}{2}\sigma_1^2 s^2 \frac{\partial^2 D}{\partial s^2} - rD = 0,$$

the standard Black-Scholes PDE.

3.2 Cross-Currency Options

This section concerns valuation of an option on an asset S_t denominated in currency F (for "foreign") which pays off in currency D (for "domestic"). We denote by f_t the exchange rate at time t, interpreted as the domestic currency price of one unit of foreign currency. Thus the currency D value of the asset S_t is $f_t S_t$ at time t. In this note we ignore interest-rate volatility and take the foreign and domestic interest rates as constants r_F, r_D respectively, so that the corresponding zero-coupon bonds have values

$$P_F(t,T) = e^{-r_F(T-t)}$$
$$P_D(t,T) = e^{-r_D(T-t)}.$$

3.2.1 Forward FX rates

To deliver one unit of currency F at time T, we can borrow $f_0 P_F(0,T)$ units of domestic currency at time 0 and buy a foreign zero-coupon bond maturing at time T. At that time the value of our short position in domestic currency is $-f_0 P_F(0,T)/P_D(0,T)$. By standard arguments, an agreement to exchange K units of domestic currency for one unit of currency F at time T is arbitrage-free if and only if $K = f_0 P_F(0,T)/P_D(0,T)$. In summary:

$$\text{Forward price} = f_0 e^{(r_D - r_F)T}.$$

This coincides with the formula for the forward price of a domestic asset with dividend yield r_F.

3.2.2 The domestic risk-neutral measure

The traded assets in the domestic economy are the domestic zero-coupon bond, value $Z_t = P_D(t,T)$, the foreign zero-coupon bond, value $Y_t = f_t P_F(t,T)$, and the foreign asset, value $X_t = f_t S_t$. An analogous set of assets is traded in the foreign economy. It is important to realize that there are *two* risk-neutral measures, depending on which economy we regard as "home".

We will assume that in the domestic risk-neutral (DRN) measure the asset S_t is log-normal, i.e. satisfies

$$dS_t = S_t\mu dt + S_t\sigma_S dw^S(t) \tag{3.23}$$

for some drift μ and volatility σ_S. The asset is assumed to have a dividend yield q. Similarly the FX rate f_t is log-normal:

$$df_t = f_t\gamma dt + f_t\sigma_f dw^f(t), \tag{3.24}$$

with drift γ and volatility σ_f. w^S and w^f are Brownian motions with $Edw^S dw^f = \rho\, dt$.

The discounted domestic value of the foreign zero-coupon bond is

$$e^{-r_D t} f_t P_F(t,T) = e^{-r_F T} f_t e^{-(r_D - r_F)t}.$$

This is a martingale in the DRN measure, which is true if and only if

$$\gamma = r_D - r_F. \tag{3.25}$$

Now consider a self-financing portfolio of foreign assets in which we hold ϕ_t units of asset S_t and keep the remaining value in foreign zero-coupon bonds. The portfolio value process V_t then satisfies

$$\begin{aligned} dV_t &= \phi_t dS_t + q\phi_t S_t dt + (V_t - \phi_t S_t) r_F dt. \\ &= V_t r_F dt + \phi_t S_t(\mu + q - r_F)dt + \phi_t S_t \sigma_S dw_t^S. \end{aligned}$$

Using (3.24),(3.25) and the Ito formula we find that the domestic value $U_t = f_t V_t$ of this portfolio satisfies

$$dU_t = r_D U_t dt + \sigma_f U_t dw_t^f + \phi_t f_t S_t \sigma_S dw_t^S + \phi_t f_t S_t(\mu + q - r_F + \rho\sigma_S\sigma_f)dt.$$

Again, the discounted value $e^{-r_D t}U_t$ is a martingale in the DRN measure, and this holds if and only if

$$\mu = r_F - q - \rho\sigma_S\sigma_f. \tag{3.26}$$

In summary, under the DRN measure the FX rate and asset value satisfy the following equations

$$df_t = f_t(r_D - r_F)dt + f_t\sigma_f dw^f(t) \tag{3.27}$$

$$dS_t = S_t(r_F - q - \rho\sigma_S\sigma_f)dt + S_t\sigma_S dw^S(t) \tag{3.28}$$

By applying the Ito formula to (3.27),(3.28) we find that $X_t := S_t f_t$, the asset price expressed in domestic currency, satisfies

$$dX_t = X_t(r_D - q)dt + X_t(\sigma_S dw^S(t) + \sigma_f dw^f(t)). \tag{3.29}$$

By computing variances we find that

$$\sigma_S w^S(t) + \sigma_f w^f(t) = \tilde{\sigma} w(t), \tag{3.30}$$

where $w(t)$ is a standard Brownian motion and

$$\tilde{\sigma} = \sqrt{\sigma_S^2 + \sigma_f^2 + 2\rho\sigma_S\sigma_f} \tag{3.31}$$

$$E[dwdw^f] = \frac{1}{\tilde{\sigma}}(\sigma_f + \rho\sigma_S). \tag{3.32}$$

Thus (3.29) becomes

$$dX_t = X_t(r_D - q)dt + X_t\tilde{\sigma} dw(t). \tag{3.33}$$

3.2.3 Option Valuation

Options on Foreign Assets

This refers to, for example, a call option with value at maturity time T

$$\max[X_T - K, 0],$$

i.e. the foreign asset value is converted to domestic currency at the spot FX rate f_T and compared to a domestically-quoted strike K. Since X_t satisfies (3.33) we see that the option value is just the Black-Scholes value for a domestic asset with volatility $\tilde{\sigma}$ given by (3.32).[1]

Currency-Protected (Quanto) Options

Here the option value at maturity is $A_0 \max[S_T - K, 0]$ units of domestic currency, where A_0 is an arbitrary exchange factor, for example the time-zero exchange rate. The option value at time zero is

$$A_0 e^{-r_D T} E(\max[S_T - K, 0]).$$

The expectation is taken under the DRN measure, in which S_t satisfies (3.28). Note that the volatility is σ_S and the drift is $r_F - q - \rho\sigma_S\sigma_f = r_D - (q + r_D - r_F + \rho\sigma_S\sigma_f)$. We can therefore calculate the option value in two equivalent ways:

(i) Use the "forward" form of the BS formula with forward $F_T = S_0 \exp((r_F - q - \rho\sigma_S\sigma_f)T)$ and discount factor $\exp(-r_D T)$.

(ii) Use the "stock" form of BS with riskless rate r_D and dividend yield $q + r_D - r_F + \rho\sigma_S\sigma_f$.

3.2.4 Hedging Quanto Options

Deriving the Hedge

The value of the quanto option given above has the usual interpretation as the initial endowment of a perfect hedging portfolio, but the formula does not indicate how the hedging takes place. To discover this, we re-derive the formula using the traditional Black-Scholes perfect hedging argument. For this we use the "objective" probability measure - not the risk-neutral measure - under which $X_t = S_t f_t$ and f_t are log-normal processes satisfying

$$dX_t = \lambda X_t dt + \tilde{\sigma} X_t d\tilde{w}(t) \tag{3.34}$$

$$df_t = \upsilon f_t dt + \sigma_f f_t dw^f(t) \tag{3.35}$$

for some drift coefficients λ, υ the value of which, it turns out, we do not need to know. The point about the hedging argument is that from the perspective of a domestic investor, S_t itself is not a traded asset: the traded assets are X_t (the domestic value of S_t) and the foreign and domestic bonds Y_t, Z_t. From (3.35), the equation satisfied by Y_t is

$$dY_t = (\upsilon + r_F)Y_t dt + \sigma_f Y_t dw^f. \tag{3.36}$$

We know from section 3.2.3 that the quanto call option value at time t is a function $C(t, S_t) = C(t, X_t/f_t)$ but we need to regard it as a function of X_t, f_t separately for hedging purposes. Note that if we define $g(t, x, f) := C(t, x/f)$ then with $C' = \partial C/\partial S$ we have

[1] Note that the sign of ρ would be reversed if we had written the FX model in terms of $1/f_t$ rather than f_t.

$$\begin{array}{llllll} \frac{\partial g}{\partial t} &= \frac{\partial C}{\partial t} & \frac{\partial g}{\partial x} &= \frac{1}{f}C' & \frac{\partial g}{\partial f} &= -\frac{x}{f^2}C' \\ \frac{\partial^2 g}{\partial x^2} &= \frac{1}{f^2}C'' & \frac{\partial^2 g}{\partial f^2} &= \frac{2x}{f^3}C' + \frac{x^2}{f^4}C'' & \frac{\partial^2 g}{\partial x \partial f} &= -\frac{1}{f^2}C' - \frac{x}{f^3}C'' \end{array} \tag{3.37}$$

Let $C(t, S_t)$ be the call value and consider the portfolio

$$V_t = C(t, X_t/f_t) - \phi_t X_t - \psi_t Y_t - \chi_t Z_t, \tag{3.38}$$

where ϕ_t, ψ_t, χ_t are the number of units of X_t, Y_t, Z_t respectively in the putative hedging portfolio. Recall that S (and hence X) pays dividends at rate q. Applying the Ito formula using (3.37) and (3.34),(3.35) and then substituting $S = X/f$ we eventually obtain

$$\begin{aligned} dV_t = &\left(\frac{\partial C}{\partial t} - SC'(v + \rho\sigma_S\sigma_f) + \frac{1}{2}\sigma_S^2 S^2 C'' - \psi(v + r_F)Y_t - \chi r_D Z_t\right) dt \\ &+ (\frac{1}{f}C' - \phi)dX - \phi q X dt - (\psi Y_t + SC')\sigma_f dw^f \end{aligned}$$

Taking

$$\phi = \frac{1}{f}C' \tag{3.39}$$

$$\psi = -\frac{1}{Y}SC' \tag{3.40}$$

this becomes

$$dV_t = \left(\frac{\partial C}{\partial t} + SC'(r_F - y - \rho\sigma_S\sigma_f) + \frac{1}{2}\sigma_S^2 s^2 C'' - \chi r_D Z_t\right) dt \tag{3.41}$$

The usual no-arbitrage argument implies that V_t must grow at the domestic riskless rate, i.e.

$$\begin{aligned} dV_t &= V_t r_D dt \\ &= (C - \frac{C'}{f}X + \frac{SC'}{Y}Y - \chi Z)r_D dt \\ &= (C - \chi Z)r_D dt \end{aligned} \tag{3.42}$$

and, from (3.41) and (3.42), this equality is satisfied if C satisfies

$$\frac{\partial C}{\partial t} + SC'(r_F - q - \rho\sigma_S\sigma_f) + \frac{1}{2}\sigma_S^2 S^2 C'' - r_D C = 0. \tag{3.43}$$

The boundary condition is

$$C(T, s) = A_0[s - K]^+ \tag{3.44}$$

If we write

$$r_F - q - \rho\sigma_S\sigma_f = r_D - (q + r_D - r_F + \rho\sigma_S\sigma_f),$$

we can see that (3.43),(3.44) is just A_0 times the Black-Scholes PDE with volatility σ_S, riskless rate r_D and dividend yield $(q + r_D - r_F + \rho\sigma_S\sigma_f)$. This agrees with the valuation obtained in Section 3.2.3.

We still have to check the two key properties of the hedging portfolio, namely perfect replication and self-financing. The former is obtained by suitably defining χ_t; from (3.42), $V_t \equiv 0$ if

$$\chi_t = \frac{C(t, S_t)}{Z_t}. \tag{3.45}$$

To check the latter, note that the hedging portfolio value is $W = \phi X + \psi Y + \chi Z$ and we now know that this is equal to the option value $C(t, S_t)$. Hence

$$\begin{aligned} dW &= dC \\ &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 C''\right) dt + C' dS \\ &= r_D C dt - SC'(r_F - q - \rho\sigma_S\sigma_f)dt + C'dS. \end{aligned} \tag{3.46}$$

(The second line is just an application of the Ito formula and the third uses the Black-Scholes PDE (3.43).) Now $S = X/f$, so using (3.34), (3.35) we get from the Ito formula

$$dS = \frac{1}{f}dX - S\frac{df}{f} - \rho\sigma_f\sigma_S S dt.$$

Thus

$$\begin{aligned} dW &= r_D C dt - SC'(r_F - q - \rho\sigma_S\sigma_f)dt + \frac{C'}{f}dX - \frac{C'S}{f}df - \rho\sigma_S\sigma_f C' S dt \\ &= r_D C dt - SC'\left(r_F dt + \frac{df}{f}\right) + qSC'dt + \frac{C'}{f}dX \end{aligned} \tag{3.47}$$

Using the definitions of ϕ, ψ and χ at (3.39),(3.40),(3.45) we see that the first, third and fourth terms of (3.47) are equal to $\chi dZ, \phi qX dt$ and ϕdX respectively. Now $Y_t = e^{-r_F(T-t)}f_t$, so

$$\begin{aligned} dY_t &= r_F Y_t dt + e^{-r_D(T-t)}df_t \\ &= r_F Y_t dt + Y_t \frac{df}{f}, \end{aligned}$$

showing that the second term in (3.47) is equal to ψdY. Thus (3.47) is equivalent to

$$dW = \phi dX + q\phi X dt + \psi dY + \chi dZ,$$

which is the self-financing property.

Interpretation of the Hedging Strategy

Recall that the hedging portfolio is

$$\phi X + \psi Y + \chi Z$$

where

$$\begin{aligned} \phi &= \frac{1}{f}C' \\ \psi &= -\frac{SC'}{Y} \\ \chi &= \frac{C}{Z} \end{aligned}$$

The net value of the first two terms is zero, and this is what eliminates the FX exposure: ϕ represents a conventional delta-hedge in Currency F, financed by Currency F borrowing (this is ψ). All increments in the hedge value are immediately "repatriated" and deposited in the home currency riskless bond Z. The value in this domestic account is $\chi Z = C$ so that, in particular, the value at the exercise time T is (for a call option) $A_0[S_T - K]^+$, the exercise value of the option.

3.3 Numéraire pairs and change of numéraire

There is a more sophisticated way of looking at the Magrabe formula, which provides extra insight. The technique involved—change of numéraire—is needed later anyway: it plays an essential role in modelling of interest rate derivatives.

Recall from section 3.1.2 that in the Margrabe problem the traded assets are S_1, S_2 and the zero-coupon bond $p(t,T)$. A self-financing portfolio can be constructed by specifying a 2-vector process α_t with the interpretation that α_t^i is the number of units of asset S_i held at time t, $i = 1, 2$. The remaining wealth is held in the zero coupon bond. The evolution of the portfolio value X_t is then

$$dX_t = \alpha_t^1 dS_1(t) + \alpha_t^2 dS_2(t) + \frac{X_t - \alpha_t^1 S_1(t) - \alpha_t^2 S_2(t)}{p(t,T)} dp(t,T).$$

Bearing in mind that $dp/p = r\,dt$, we find that in discounted units $\tilde{S}_i(t) = e^{-rt} S_i(t), i = 1, 2$ and $\tilde{X}_t = e^{-rt} X_t$, this portfolio equation becomes

$$d\tilde{X}_t = \alpha_t^1 d\tilde{S}_1(t) + \alpha_t^2 d\tilde{S}_2(t).$$

Under the risk-neutral measure P, the discounted prices $\tilde{S}_i$ are martingales, and hence $\tilde{X}_t$ is a P (local) martingale, for any trading strategy α. Now define $B(t) = e^{rt}$. This is the portfolio value corresponding to the strategy $\alpha^1 = \alpha^2 \equiv 0,\ \alpha_t^3 = 1/p(0,T)$ (with initial investment 1, buy $1/p(0,T)$ units of zero-coupon bond at time 0 and hold until time T). Then we can write the discounted prices as $\tilde{S}_i(t) = S_i(t)/B(t), \tilde{X}_t = X_t/B(t)$, i.e. the discounted prices are the prices expressed in units of the *numéraire asset* $B(t)$. The measure P is the unique measure such that these price ratios are all local martingales.

In fact, using $B(t)$ as numéraire asset is an *arbitrary choice*. Rather than describing P as *the* risk-neutral measure, one should say that (B, P) is a 'numéraire pair'. In the next section we describe this idea in a somewhat more general setting. We will return to Margrabe in section 3.3.3 below.

3.3.1 Numéraire pairs

Let $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a stochastic basis and $(S_0(t), \ldots, S_n(t))$ be $\mathcal{F}_t$-adapted continuous semimartingales. We assume that $S_0(t) > 0$ for all $t \in [0,T]$ a.s. and (normalizing if necessary) that $S_0(0) = 1$. These are the prices of $n+1$ traded assets. A *trading strategy* is an n-vector process $\alpha_t = (\alpha_t^1, \ldots, \alpha_t^n)$, where each component α_t^i is an adapted process such that $\int_0^T \alpha_t^i d[S_i]_t < \infty$ a.s. α_t^i is the number of units of asset i in the portfolio at time t. We will call X_t a *portfolio process* if X_t is the value at time t of the portfolio corresponding to some trading strategy. X_t satisfies the following equation.

$$dX_t = \alpha_t dS(t) + \frac{X_t - \alpha_t S(t)}{S_0(t)} dS_0(t). \tag{3.48}$$

(Here αS denotes the vector inner product. The value of the holdings in assets 1 to n is $\alpha_t S_t$ and the residual value $X_t - \alpha_t S(t)$ buys $(X_t - \alpha_t S(t))/S_0(t)$ units of asset 0.) We call S_0 the *numéraire asset.* A *numéraire pair* is a pair (S_0, Q) where Q is a probability measure, equivalent to P, such that the process $X_t/S_0(t)$ is a local martingale for all portfolio processes X_t.

Proposition 1 *Let (S_0, Q) be a numéraire pair and let X_t be the value of the self-financing portfolio corresponding to a trading strategy $\alpha = (\alpha^1, \ldots, \alpha^n)$. Then the value $\tilde{X}_t = X_t/S_0(t)$ (i.e., the value expressed in units of S_0) is given by*

$$d\tilde{X}_t = \sum_{i=1}^n \alpha_t^i d\tilde{S}_i(t), \quad \tilde{X}_0 = X_0, \tag{3.49}$$

where $\tilde{S}_i = S_i/S_0$. Hence $\tilde{X}_t$ is a Q-local martingale.

Proof. Using the Ito formula for continuous semimartingales, we have

$$d\left(\frac{1}{S_0}\right) = -\frac{1}{S_0^2}dS_0 + \frac{1}{S_0^3}d[S_0]. \tag{3.50}$$

Now, using (3.48), (3.50) and the product formula $d(XY) = XdY + YdX + d[X,Y]$ with $Y = 1/S_0$, we obtain

$$d\left(\frac{X_t}{S_0(t)}\right) = \sum_{i=1}^{n} \alpha_i \left(\frac{1}{S_0}dS_i - \frac{S_i}{S_0^2}dS_0 - \frac{1}{S_0^2}d[S_i, S_0] + \frac{S_i}{S_0^3}d[S_0]\right). \tag{3.51}$$

If we set $\alpha_k = 1$ and $\alpha_i = 0$, $i \neq k$ then $X_t = S_k(t)$ and we see that the kth term on the right of (3.51) is equal to $\alpha_k d(S_k/S_0)$. The result follows. ◇

Proposition 2 *Suppose the n-vector process $\tilde{S}_t = (\tilde{S}_t^1, \ldots, \tilde{S}_t^n)$ has the* martingale representation property*: for any $\mathcal{F}_T$ measurable random variable Y there is an n-vector integrand ϕ such that*

$$Y = E_Q[Y] + \int_0^T \phi_t d\tilde{S}_t. \tag{3.52}$$

Then the unique arbitrage-free value of a contingent claim whose exercise value is H is

$$v_0 = E_Q\left[\frac{H}{S_0(T)}\right]. \tag{3.53}$$

More generally, the value of the claim at any time $t \in [0, T]$ is

$$v_t = S_0(t) E_Q\left[\frac{H}{S_0(T)}\middle| \mathcal{F}_t\right].$$

Proof. In (3.52), take $Y = H/S_0(T)$ and interpret (3.52) as defining a trading strategy which replicated the exercise value H. The unique arbitrage-free value of the claim is then the initial endowment of the replicating strategy, which is equal to $E_Q[Y]$. The second part follows from the fact that $\tilde{v}_t = v_t/S_0(t)$ is a Q-martingale. ◇

3.3.2 Change of numéraire

Proposition 3 *Let (N_1, Q_1) be a numéraire pair and let N_2 be a second numéraire. Suppose that*

$$E_{Q_1}\left[\frac{N_2(T)}{N_1(T)}\right] = 1. \tag{3.54}$$

Then (N_2, Q_2) is a numéraire pair, where Q_2 is defined by

$$\frac{dQ_2}{dQ_1} = \frac{N_2(T)}{N_1(T)}. \tag{3.55}$$

Proof. Since (N_1, Q_1) is a numéraire pair, and using condition (3.54), $\Lambda_t \equiv N_2(t)/N_1(t)$ is a Q_1 martingale. Now a process Y_t is a Q_2 local martingale if and only if $Y_t\Lambda_t$ is a Q_1 local martingale. Taking $Y_t = X_t/N_2(t)$ for an arbitrary portfolio process X_t we have $Y_t\Lambda_t = X_t/N_1(t)$ which is indeed a Q_1 local martingale, so $X_t/N_2(t)$ is a Q_2 local martingale. Hence (N_2, Q_2) is a numéraire pair. ◇

This result shows that once we have obtained a martingale measure Q_1 corresponding to some particular numéraire N_1, we can switch at will to essentially any other numéraire N_2, for which a martingale measure will be given by the formula (3.55). Which numéraire we should use in a particular case is purely a matter of convenience. The exchange option problem is a good illustration.

3.3.3 Margrabe revisited

The exercise value of the Margrabe exchange option is $H = [S_1(T) - S_2(T)]^+$. Let us take $N_2(t) = S_2(t)/s_2$ as numéraire and denote by Q_2 the corresponding martingale measure (so that (N, Q_2) is a numéraire pair). Then from (3.53) the Margrabe value is

$$\begin{aligned} v_0 &= E_{Q_2}\left[\frac{s_2}{S_2(T)}[S_1(T) - S_2(T)]^+\right] \\ &= s_2 E_{Q_2}[Y_T - 1]^+, \end{aligned}$$

where $Y_t = S_1(t)/S_2(t) = s_2 S_1(t)/N_t$. By the definition of numéraire pair we know that Y_t is a Q_2-local martingale, so Q_2 must be a measure under which Y_t has zero drift. But then we immediately obtain

$$dY = Y\sigma dw,$$

where σ is given by (3.7) and w is a Q_2-Brownian motion. (To identify σ, it is only necessary to compute the quadratic variation of Y, which is the same under all equivalent measures. The calculations in Section 3.1.1 show what this is.) We have thus recovered the result of Section 3.1.1, namely that v_0 is s_2 times the value of a call option on Y with strike 1.

The martingale measure Q_2 in this section is the same as the measure $\tilde{P}$ of Section 3.1.1. We introduced it there as a computational device and it was not at all clear why Y_t turned out to be a $\tilde{P}$ martingale. Now we know why. In 3.1.1, P is the measure corresponding to the numéraire $N_1(t) = e^{rt}$. The reader can check that the Girsanov exponential (3.14) coincides with the numéraire change formula (3.55).

4

Fixed Income

4.1 Bonds: the basics

4.1.1 The price/yield relationship

A bondholder receives interest payments or *coupons* on fixed dates at regular intervals (say, every 6 months) and at the final maturity date receives the final coupon plus the *par value*, which we will normalize as 1.

The coupon payments are specified by a *rate* (5.5%,..), a *frequency* (1,2,4: the number of payments per year) and a *basis* stating how the *accrual* or *day count* is calculated. Typical bases are actual/actual, actual/365, 30/360.. For example if we have a rate of 5.5% paid semi-annually (frequency = 2) on an actual/365 basis then the payment dates are 6 months apart and the coupon payment on a particular payment date is $(d/365) * 0.055$, where d is the number of days since the last coupon date. The accrual factor $(d/365)$ is very nearly, but not exactly, equal to 1/2. The first coupon is paid 6 months after the bond is issued.

For simplicity, consider a bond with frequency 1, coupon c and basis actual/actual (or 30/360), so that the accrual factor is 1, and maturity n years. If the price at issue is p, the *yield* is the number y satisfying $p = B(y)$ where

$$B(y) = \sum_{i=1}^{n} \frac{c}{(1+y)^i} + \frac{1}{(1+y)^n}$$

(Interpretation: all the coupon payments could be financed by depositing at time 0 the amount p in an account paying annual interest y.) Note the inverse relationship: high yield $\Leftrightarrow$ low price. The *(modified) duration* of the bond is

$$D(y) = -\frac{1}{B(y)} \frac{dB(y)}{dy}.$$

Note that this has units of *years*. For a *zero-coupon* bond ($c = 0$) the duration is $n/(1+y) \approx n$, whereas a coupon bond has shorter duration: maybe 7 years for a 10-year bond issued at par. The *convexity* of the bond is

$$C(y) = \frac{d^2 B(y)}{dy^2}.$$

To illustrate the effect of convexity, suppose that the yield is a random variable Y with expected value y_0. We define the *yield volatility* as $\sigma = \sqrt{E(Y - y_0)^2}/y_0$. Then

$$B(Y) = B(y_0) + \frac{dB(y_0)}{dy}(Y - y_0) + \frac{1}{2}\frac{d^2 B(y_0)}{dy^2}(Y - y_0)^2 + \cdots$$

so that to second order

$$EB(Y) = B(y_0) + \frac{1}{2}C(y_0)y_0^2\sigma^2.$$

Yield volatility increases the expected bond value due to the convexity effect.

4.1.2 Floating rate notes

Suppose an annual interest rate L_i is set at the beginning of year $i, i = 1, 2, \ldots$, so that \$1 at the beginning of year i becomes \$$(1 + L_i)$ at the end of the year. The corresponding discount factor is $1/(1 + L_i)$. A floating rate note is an n-year bond whose coupon c_i paid at the end of year i is equal to L_i. A key fact is the following: *at coupon dates, a floating rate note is always at par*. You can think of this as a consequence of the identity

$$\sum_{i=1}^{n} \frac{L_i}{\prod_{j=1}^{i}(1 + L_j)} + \frac{1}{\prod_{j=1}^{n}(1 + L_j)} = 1.$$

In general the rate L_i is a random variable whose value is not known until the beginning of year i. We need extra information to be able to value a *fixed-coupon* bond.

4.2 A general valuation model

All processes are assumed to be $\mathcal{F}_t$-adapted on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ where P will be the unique risk-neutral measure. There are traded assets with price processes $S_i(t)$. The holder of asset i receives cumulative dividends $D_i(t)$, i.e. the dividend received in the time interval $]t, t + dt]$ is $dD_i(t) = D_i(t + dt) - D_i(t)$. There is a savings account paying interest at continuously-compounde rate $r(t)$, again an adapted random process. A self-financing portfolio with trading strategy ϕ (i.e. $\phi_i(t)$ is the number of units of asset i in the portfolio at time t) thus has wealth process X_t satisfying

$$dX_t = \sum_i \phi_i(dS_i + dD_i) + \left(X_t - \sum_j \phi_j S_j\right) r(t)\,dt. \tag{4.1}$$

Define

$$B(t) = \exp\left(\int_0^t r(s)ds\right).$$

Then

$$\begin{aligned} d\left(B^{-1}(t)X_t\right) &= \sum_i B^{-1}\phi_i(dS_i - rS_i dt + dD_i) \\ &= \sum_i \phi_i d(B^{-1}S_i) + \sum_i \phi_i B^{-1}dD_i). \end{aligned} \tag{4.2}$$

In particular, taking $\phi_i(t) \equiv 1$ for $i = j$ and $\phi_i(t) \equiv 0$ otherwise, we obtain

$$B^{-1}(t)X_t = B^{-1}(t)S_j(t) + \int_0^t B^{-1}(s)dD_j(s) =: M_j(t). \tag{4.3}$$

If P is the risk-netral measure corresponding to the savings account numéraire $B(t)$ then $M_j(t)$ *is a martingale for each* j. We then see that (4.2) can be written

$$d\left(B^{-1}(t)X_t\right) = \sum_i \phi_i(t)dM_i(t). \tag{4.4}$$

Thus if Z is an $\mathcal{F}_t$-measurable random variable such that $Z = X_t$ a.s. for some self-financing strategy ϕ then – assuming the stochastic integrals in (4.4) are true martingales – the value of Z at time 0 is

$$E\left[e^{-\int_0^t r(u)du} Z\right],$$

and more generally the value at some intermediate time $s \in [0,t]$ is

$$E\left[e^{-\int_s^t r(u)du} Z \middle| \mathcal{F}_s\right]. \tag{4.5}$$

The market is complete if, for each $t \in [0,T]$, *every* contingent claim Z (in some class about which we will not be too precise) with exercise time t can be replicated, i.e. $Z = X_t$ for some trading strategy ϕ.

A special but exceptionally important case is the "contingent claim" $Z \equiv 1$. Then (4.5) gives us the time-s value $p(s,t)$ of a zero-coupon (ZC) bond, paying \$1 at time t, as

$$p(s,t) = E\left[e^{-\int_s^t r(u)du} \middle| \mathcal{F}_s\right] = B(s)E[B^{-1}(t)|\mathcal{F}_s] \tag{4.6}$$

4.3 Interest rate contracts

4.3.1 Libor rates

A zero-coupon (ZC) bond value $p(s,t)$ is equivalent to a simple interest payment for the period $[s,t]$ of L satisfying

$$p(s,t) = \frac{1}{1+\theta_{st}L},$$

or equivalently

$$L = \frac{1}{\theta_{st}}\left(\frac{1}{p(s,t)} - 1\right), \tag{4.7}$$

where θ_{st} is the accrual factor (in the appropriate basis) for the interval $[s,t]$. L defined by (4.7) is the *Libor rate*. Note that

- L is set at time s (i.e. it is $\mathcal{F}_s$ measurable) but paid at time t.
- The value at time s of the Libor payment at time t is

$$\begin{aligned} E\left[e^{-\int_s^t r(u)du}\theta_{st}L \middle| \mathcal{F}_s\right] &= p(s,t)\theta_{st}L \\ &= 1 - p(s,t) \end{aligned}$$

Because of the latter fact, the accrual factor plays no role in the theory. It is just a conventional way of specifying what the Libor *rate* is, while the actual *payment* depends only on the ZC bond values.

For $t < T_1 < T_2$ the *forward bond* $p^f(t;T_1,T_2)$ and *forward Libor rate* $L^f(t;T_1,T_2)$ at t for the period $[T_1,T_2]$ are

$$p^f(t;T_1,T_2) = \frac{p(t,T_2)}{p(t,T_1)}$$

and

$$\begin{aligned} L^f(t;T_1,T_2) &= \frac{1}{\theta_{T_1T_2}}\left(\frac{1}{p^f(t;T_1,T_2)} - 1\right), \\ &= \frac{1}{\theta_{T_1T_2}}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right) \end{aligned}$$

Suppose that, at time t, I agree to make a Libor payment at time T_2 (with rate set at time T_1) in exchange for a payment at a rate K fixed now, at time t; this is a *forward rate agreement (FRA)*. Fact: *the unique arbitrage-free value of the fixed side in a FRA is $K = L^f$, the forward Libor rate.* Indeed, the corresponding hedging strategy is as follows: at time t,

- borrow a number $\theta_{T_1T_2}L^f$ of T_2-ZC bonds, value $p(t,T_2)(p(t,T_1)/p(t,T_2) - 1) = p(t,T_1) - p(t,T_2)$; the fixed payment θL^f at time T_2 exactly redeems these bonds.
- buy one T_1-ZC bond and sell one T_2-ZC bond.
- at time T_1, these bonds have value $1-p(T_1,T_2)$, enough to buy a number $(1-p(T_1,T_2))/p(T_1,T_2)$ of T_2-ZC bonds. At time T_2 these have value $(1/p(T_1,T_2) - 1) = \theta_{T_1T_2}L$.

4.3.2 Swap rates

An interest rate swap is specified by maturity, frequency, basis, notional amount N and fixed side rate K. On each coupon date t_i one party (the 'fixed side') pays $N\theta_i K$ while the other (the 'floating side') pays $N\theta L_i$ where L_i is the Libor rate set at t_{i-1}. Here $\theta_i = \theta_{t_1t_2}$. We will take $N = 1$ henceforth.

Fictitiously adjoin to the swap equal and opposite payments of 1 at the maturity date. Then the floating side is equivalent to a floating rate note, with value 1 at time 0, while the fixed side is equivalent to a coupon bond, with value

$$\sum_{i=1}^{n} K\theta_i p(0,t_i) + p(0,t_n).$$

The *swap rate* is the value of K such that the swap has value 0 at time 0. Clearly this value is

$$K_0 = \frac{1 - p(0,t_n)}{\sum_{i=1}^{n} \theta_i p(0,t_i)} \tag{4.8}$$

At later times this swap does not generally have value zero because the same fixed-side rate K_0 is maintained throughout. For example the value at t_j, to the party paying fixed, is

$$1 - \sum_{i=j+1}^{n} K_0\theta_i p(t_j,t_i) - p(t_j,t_n), \tag{4.9}$$

since the floating side always has value 1. The swap rate K_j at t_j is, in our model, an $\mathcal{F}_{t_j}$-measurable random variable. The *forward swap rate* at t_j is, by analogy with (4.8)

$$\begin{aligned} K_j^f &= \frac{1 - p^f(0;t_j,t_n)}{\sum_{i=j+1}^{n} \theta_i p^f(0;t_j,t_i)} \\ &= \frac{p(0,t_j) - p(0,t_n)}{\sum_{i=j+1}^{n} \theta_i p(0,t_i)} \end{aligned}$$

EXERCISE: Show that $K = K_j^f$ is the unique no-arbitrage value of an agreement, made at time 0, to enter a swap at time t_j at fixed rate K.

4.3.3 Interest rate options

The standard interest-rate options are *caps, floors* and *swaptions.* A *cap* pays a cash amount $\theta_i[L_i - K]^+$ at each coupon date $i, i = 1 \ldots n$. In view of (4.7) we have

$$\theta_i[L_i - K]^+ = \left[\frac{1}{p(t_{i-1},t_i)} - (1+\theta_i K)\right]^+$$

and the value of this payment at time t_{i-1} is

$$p(t_{i-1}, t_i)\left[\frac{1}{p(t_{i-1}, t_i)} - (1+\theta_i K)\right]^+ = (1+\theta_i K)[\kappa_i - p(t_{i-1}, t_i)]^+,$$

where $\kappa_i = 1/(1+\theta_i K)$. Thus a cap is equivalent to a series of *caplets*, each caplet being equivalent to a put option on the ZC bond. In our model the caplet value is

$$\frac{1}{\kappa_i} E\left(e^{-\int_0^{t_{i-1}} r(s)ds}[\kappa_i - p(t_{i-1}, t_i)]^+\right)$$

A *floor* pays $\theta_i[K - L_i]^+$.

A *swaption* is the right to enter a swap at a fixed-side rate K, starting at a time t_j in the future. It is a 'payer's swaption' if the holder will enter the swap paying the fixed side, and a 'receiver's swaption' otherwise. From (4.9) the value of a payer's swaption with strike K is

$$E\left(e^{-\int_0^{t_j} r(s)ds}[1 - \sum_{i=j+1}^{n} K\theta_i p(t_j, t_i) - p(t_j, t_n)]^+\right).$$

It is equivalent to a put option on a coupon bond, with coupon K, with strike 1.

4.3.4 Futures

Very briefly, a futures contract maturing at time T on an asset S_i is a traded asset with 'price' F_t such that

- The futures contract can be entered at zero cost at any time;
- A holder of the contract receives a payment $F_{t+dt} - F_t$ in the interval $[t, t+dt]$.
- At maturity T, $F_T = S_i(T)$.

From this description it is clear that the futures 'price' is not a price at all. It is a dividend. The future is an asset S_j with price $S_j(t) \equiv 0$ and dividend process $D_j(t) = F_t$. In view of (4.3) we see that

$$M_j(t) = \int_0^t B^{-1}(s)dF_s$$

is a martingale, so that

$$F_t = \int_0^t B(s)dM_j(s)$$

is a martingale. Since $F_T = S_i(T)$, this shows that

$$F_t = E[S_i(T)|\mathcal{F}_t], \qquad t \le T.$$

Recall that the *forward price* G_t is the no-arbitrage exchange price for $S_i(T)$ fixed at time t, i.e. G_t satisfies

$$E\left[e^{-\int_t^T r(s)ds}(G_t - S_i(T))\middle|\mathcal{F}_t\right] = 0.$$

Hence

$$G_t = \frac{1}{p(t,T)} E\left[e^{-\int_t^T r(s)ds} S_i(T)\middle|\mathcal{F}_t\right].$$

The difference between forward and futures prices is therefore

$$\begin{aligned} F_t - G_t &= E[S_i(T)|\mathcal{F}_t] - \frac{1}{p(t,T)} E\left[e^{-\int_t^T r(s)ds} S_i(T) \middle| \mathcal{F}_t \right] \\ &= \frac{1}{p(t,T)} E\left[S_i(T)(p(t,T) - e^{-\int_t^T r(s)ds}) \middle| \mathcal{F}_t \right] \\ &= \frac{-1}{p(t,T)} \mathrm{cov}_{\mathcal{F}_t}\left(S_i(T), e^{-\int_t^T r(s)ds} \right), \end{aligned} \tag{4.10}$$

where $\mathrm{cov}_{\mathcal{F}_t}(X, Y)$ denotes the conditional covariance of X and Y. In particular, *forward and future are the same if there is no interest-rate volatility.*

Exchange-traded futures include the *Eurodollar futures contract*, whose settlement value at time T is $100(1 - L)$, where L is the 3-month Libor rate set at T. (The reason for this convention is to maintain the 'high rate $\Leftrightarrow$ low price' relationship, as for bonds.) It is important to note that a futures price of, say, 94.5 does not mean that forward Libor is 5.5%: this figure has to be adjusted by the 'convexity correction' (4.10). Note that when S_i in (4.10) is a Libor rate, it is generally positively correlated with $r(s)$ and therefore negatively correlated with $e^{-\int_t^T r(s)ds}$. Thus the right-hand side of (4.10) is positive, so the futures price is *bigger* than the forward price.

4.4 Pricing interest-rate options

The standard market convention for pricing plain-vanilla interest-rate options is to use the Black 'forward' formula

$$p(0,T)[FN(d_1) - KN(d_2)], \tag{4.11}$$

where d_1, d_2 are the usual volatility-related factors. This can be applied to caplets, with F as the forward Libor rate, or to swaps with F as the forward swap rate. There is some apparent inconsistency with this approach: the whole point is that interest rates in the future are random, but we treat the discount factor $p(0,T)$ in (4.11) as deterministic. In this section we show that something close to this approach is in fact consistent if we re-interpret things in terms of 'forward measures'. A good reference for this material is Hunt and Kennedy [4].

4.4.1 The forward measure

In the framework of Section 4.2, the *forward price* $F_i(t,T)$ of a traded asset S_i is the price agreed at time t for exchange at time T, i.e. the value of κ such that

$$E\left[e^{-\int_t^T r(u)du} (\kappa - S_i(T)) \middle| \mathcal{F}_t \right] = 0,$$

or equivalently

$$\kappa p(t,T) = E\left[e^{-\int_t^T r(u)du} S_i(T) \middle| \mathcal{F}_t \right]. \tag{4.12}$$

Since $M_i(t)$ given by (4.3) is a martingale, we see that

$$F_i(t,T) = \frac{1}{p(t,T)} S_i(t) - \frac{1}{p(t,T)} E\left[\int_t^T e^{-\int_t^s r(u)du} dD_i(s) \middle| \mathcal{F}_t \right]. \tag{4.13}$$

The value $C(t)$ at time t of an option on S_i, maturing at T with exercise value $h(S_i(T))$ is, as usual,

$$C(t) = E\left[e^{-\int_t^T r(u)du} h(S_i(T)) \middle| \mathcal{F}_t \right] \tag{4.14}$$

DEFINITION: The T-*forward measure* on $(\Omega, \mathcal{F}_T)$ is the measure P^T defined by the Radon-Nikodým derivative

$$\frac{dP^T}{dP} = \frac{e^{-\int_0^T r(u)du}}{p(0,T)} = \frac{\frac{1}{p(0,T)}}{B(T)}. \tag{4.15}$$

Note that P^T is well-defined in that the right hand side of (4.15) is strictly positive and has expectation 1. We see from (4.15) and the general change-of-numéraire formula that P^T is the risk-neutral measure corresponding to a numéraire $N(t)$ where $N(T) = 1/p(0,T)$. Since $N(t)/B(t)$ is a P-martingale, this implies that $N(t) = p(t,T)/p(0,T)$. Thus *moving to the T-forward measure is equivalent to changing the numéraire from the savings account $B(t)$ to the zero-coupon bond $p(t,T)/p(0,T)$.*

By the standard formula for conditional expectation under change of measure,

$$\begin{aligned} E^T[h(S_i(T))|\mathcal{F}_t] &= \frac{E\left[e^{-\int_0^T r(u)du} h(S_i(T)) \middle| \mathcal{F}_t\right]}{E\left[e^{-\int_0^T r(u)du} \middle| \mathcal{F}_t\right]} \\ &= \frac{E\left[e^{-\int_t^T r(u)du} h(S_i(T)) \middle| \mathcal{F}_t\right]}{p(t,T)}, \end{aligned}$$

so that

$$C(t) = p(t,T)E^T[h(S_i(T))|\mathcal{F}_t].$$

The key fact about the forward measure is this:

Proposition 4.1. *The forward price is a martingale under the T-forward measure.*

PROOF: Indeed, for $s < t$, we have from (4.12)

$$\begin{aligned} E^T[F_i(t,T)|\mathcal{F}_s] &= \frac{E\left[e^{-\int_0^T r(u)du} \frac{E\left[e^{-\int_t^T r(u)du} S_i(T) \middle| \mathcal{F}_t\right]}{p(t,T)} \middle| \mathcal{F}_s\right]}{E\left[e^{-\int_0^T r(u)du} \middle| \mathcal{F}_s\right]} \\ &= \frac{E\left[e^{-\int_s^t r(u)du} E\left[e^{-\int_t^T r(u)du} S_i(T) \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_s\right]}{p(s,T)} \\ &= \frac{E\left[e^{-\int_s^T r(u)du} S_i(T) \middle| \mathcal{F}_s\right]}{p(s,T)} = F_i(s,T). \end{aligned}$$

This is the martingale property. $\diamondsuit$

Proposition 4.1 implies in particular that

$$\begin{aligned} E^T[S_i(T)] &= E^T[F(T,T)] \\ &= F(0,T), \end{aligned} \tag{4.16}$$

where $F(0,T)$ is given by (4.13) with $t = 0$. This gives us our first pricing formula.

Proposition 4.2. *Suppose $S_i(t)$ is log-normally distributed in the T-forward measure, with volatility σ. Then the no-arbitrage price at time 0 of a call option with exercise value $[S_i(T)-K]^+$ is given by the Black formula*

$$C(0) = p(0,T)[F(0,T)N(d_1) - KN(d_2)], \tag{4.17}$$

where

$$d_1 = \frac{\log(F(0,T)/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

PROOF: In view of (4.16) the price $S_i(T)$ is given by

$$S_i(T) = F(0,T)\exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}X\right)$$

where $X \sim N(0,1)$. The result follows by standard calculations. $\diamondsuit$

4.4.2 Forwards and futures

In section 4.3.4 we showed that the futures price is a martingale in the risk-neutral measure, whereas Proposition 4.1 shows that the forward is a martingale in the T-forward measure. Thus the convexity correction (4.10) is equal to the difference in expected value under the two measures, $E[S_i(T)] - E^T[S_i(T)]$.

4.4.3 Caplets

Consider a caplet where the Libor rate is set at T_1 and paid at T_2. Let θ be the accrual factor. Then the forward Libor rate at $t \le T_1$ is

$$L_t^f = \frac{1}{\theta}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right).$$

In this case *the forward Libor rate is a martingale in the T_2-forward measure.* Indeed for $s < t$

$$\begin{aligned} E^{T_2}[L_t^f|\mathcal{F}_s] &= \frac{E\left[e^{-\int_0^{T_2} r(u)du}\frac{1}{\theta}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right)|\mathcal{F}_s\right]}{E\left[e^{-\int_0^{T_2} r(u)du}|\mathcal{F}_s\right]} \\ &= \frac{E\left[e^{-\int_s^{t} r(u)du}p(t,T_2)\frac{1}{\theta}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right)|\mathcal{F}_s\right]}{p(s,T_2)} \\ &= \frac{1}{\theta}\frac{p(s,T_1) - p(s,T_2)}{p(s,T_2)} = L_s^f. \end{aligned}$$

Thus, as in Proposition 4.2, if the Libor rate is assumed to be log-normally distributed in the T_2-forward measure we can use the Black formula (4.17) to price the caplet, slightly modified because of the different setting and paying times. Specifically, the price is

$$p(0,T_2)[L_0^f N(d_1) - KN(d_2)]$$

with $T := T_1$ in d_1, d_2.

4.4.4 Swaptions

Here we have to be a little more ingenious. As discussed in section 4.3.3, the value at time t of the right to enter a swap at time $t_0 > t$ at fixed-side rate K is

$$SV_t = E\left\{ e^{-\int_t^{t_0} r(u)du} \left[1 - K\sum_{i=1}^{n} \theta_i p(t_0, t_i) - p(t_0, t_n)\right]^+ \middle| \mathcal{F}_t \right\}, \tag{4.18}$$

where the swap coupon dates are $t_1, \ldots, t_n$ and θ_i are the accrual factors. The forward swap rate is

$$F_t = \frac{p(t,t_0) - p(t,t_n)}{\sum_i \theta_i p(t,t_i)} = \frac{p(t,t_0) - p(t,t_n)}{p_A(t)},$$

where $p_A(t) = \sum_i \theta_i p(t,t_i)$ known as the 'present value of a basis point' (it is the value at time t of unit payments received at $t_1, \ldots, t_n$).

The swaption value (4.18) can be written

$$\begin{aligned} SV_t &= E\left\{ e^{-\int_t^{t_0} r(u)du} \sum_{i=1}^{n} \theta_i p(t_0,t_i)[F_{t_0} - K]^+ \middle| \mathcal{F}_t \right\} \\ &= E\left\{ \sum_i \theta_i e^{-\int_t^{t_i} r(u)du} [F_{t_0} - K]^+ \middle| \mathcal{F}_t \right\}. \end{aligned} \tag{4.19}$$

Now define the *annuity measure* P_A as

$$\frac{dP_A}{dP} = \frac{\sum_i \theta_i e^{-\int_0^{t_i} r(u)du}}{p_A(0)}. \tag{4.20}$$

The swaption value is then expressed in terms of the annuity measure as

$$SV_t = p_A(t) E_A([F_{t_0} - K]^+ | \mathcal{F}_t). \tag{4.21}$$

Expression (4.21) shows that a payer's swaption is equivalent to a call option on the swap rate.

Proposition 4.3. *The forward swap rate F_t is a martingale in the annuity measure, on the interval $t \in [0, t_0]$.*

PROOF: Exercise! (The calculation is very similar to the forward Libor rate case.)

This gives us the Black formula for pricing swaptions. Assume that the swap rate F_{t_0} is log-normal in the annuity measure. In view of Proposition 4.3, $E_A[F_{t_0}] = F_0$ and the swaption price at time 0 is

$$p_A(0)[F_0 N(d1) - K N(d_2)].$$

Finally, we want to understand the change-of-numéraire aspects of the annuity measure. These are complicated by the fact that the swaption exercise value is $\mathcal{F}_{t_0}$-measurable but dP_A/dP given by (4.20) is not $\mathcal{F}_{t_0}$-measurable. The Radon-Nikodym derivative restricted to the σ-field $\mathcal{F}_{t_0}$ is just the conditional expectation

$$\begin{aligned} \left.\frac{dP_A}{dP}\right|_{\mathcal{F}_{t_0}} &= E\left[\frac{\sum \theta_i e^{-\int_0^{t_i} r(s)ds}}{p_A(0)} \middle| \mathcal{F}_{t_0} \right] \\ &= \frac{e^{-\int_0^{t_0} r(s)ds} \sum \theta_i p(t_0,t_i)}{p_A(0)} \\ &= \frac{p_A(t_0)/p_A(0)}{B(t_0)}. \end{aligned}$$

Since this process is a P-martingale, we have shown that moving to the annuity measure P_A is equivalent to a change of numéraire from $B(t)$ to the normalized annuity $p_A(t)/p_A(0)$. Thus the value at time 0 of any $\mathcal{F}_{t_0}$-measurable payment Y received at time t_0 is

$$p_A(0)E_A\left[\frac{Y}{p_A(t_0)}\right].$$

In the swaption case, $Y = p_A(t_0)[F_{t_0} - K]^+$, so the value is $Y = p_A(0)E_A\left([F_{t_0} - K]^+\right)$, as we found earlier.

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